

# Matrix Gauge Fields and Noether's Theorem

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## Preface and Summary

These notes are about systems of 1st and 2nd order (non-)linear partial differential equations which are formed from a *Lagrangian density*  $\mathbf{L}_\psi : \mathbb{R}^N \rightarrow \mathbb{C}$ ,

$$\text{Symbolically : } \underline{x} \mapsto \mathbf{L}_\psi(\underline{x}) = \mathbf{L}(\underline{\psi}(\underline{x}) ; \nabla \underline{\psi}(\underline{x}) ; \underline{x}) ,$$

by means of the usual Euler-Lagrange variational rituals. The non subscripted  $\mathbf{L}$  will denote the '*proto-Lagrangian*', which is a function of a finite number of variables:

$$\mathbf{L} : \mathbb{C}^{r \times c} \times \mathbb{C}^{Nr \times c} \times \mathbb{R}^N \rightarrow \mathbb{C} .$$

In this  $\mathbf{L}$  one has to substitute matrix-valued functions  $\underline{\psi} : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$  and  $\nabla \underline{\psi} : \mathbb{R}^N \rightarrow \mathbb{C}^{Nr \times c}$  for obtaining the Lagrangian density  $\mathbf{L}_\psi$ . In our considerations the role and the special properties of the proto-Lagrangian  $\mathbf{L}$  are crucial.

These notes have been triggered by physicist's considerations: **(1)** on obtaining the 'classical', that is the 'pre-quantized', wave equations for matter fields from variational principles, **(2)** on conservation laws and **(3)** on 'gauge field extensions'. For the humble mathematical anthropologist the rituals in physics textbooks have not much changed during the last four decades. Neither have they become much clearer. Compare e.g. [DM] and [W].

The underlying notes give special attention to the following

- In expressions (= 'equations') for Lagrange densities often both  $\underline{\psi}$  and its hermitean transposed  $\underline{\psi}^\dagger$  appear. Are they meant as independent variables or not? Mostly, from the context the suggestion arises that 'variation' of  $\underline{\psi}$  and 'variation' of  $\underline{\psi}^\dagger$  lead

to the same Euler-Lagrange equations. Why? Our remedy is doubling the matrix entries in the proto-Lagrangian and thereby making the Lagrangian density explicitly dependent on both  $\underline{\psi}, \underline{\psi}^\dagger$  and their derivatives: So for  $\mathbb{L}_\psi(\underline{x})$  we take expressions like  $\mathcal{L}_\psi(\underline{x}) = \mathcal{L}(\underline{\psi}(\underline{x}); \underline{\psi}(\underline{x})^\dagger; \nabla \underline{\psi}(\underline{x}); \nabla \underline{\psi}(\underline{x})^\dagger; \underline{x})$ . A suitable condition is then that the *Lagrangian functional*

$$\mathcal{L}[\underline{\psi}] = \int_{\mathbb{R}^N} \mathcal{L}_\psi(\underline{x}) \, d\underline{x}$$

only takes real values (Thm 2.4).

- For 'free gauge fields' the situation is somewhat different. Now the dependent variables, named  $\mathcal{A}_\mu, 1 \leq \mu \leq N$ , take their values in some fixed Lie-algebra  $\mathfrak{g} \subset \mathbb{C}^{c \times c}$ . Although  $\mathfrak{g}$  mostly contains complex matrices it is a *real vector space* in interesting cases. (Note that  $\mathfrak{u}(1) = i\mathbb{R}$  is a *real* vector space!). Therefore it needs a separate treatment.
- The traditional conservation laws for quantities like energy, momentum, moment of momentum, ..., turn out to be based on *External Infinitesimal Symmetries* of the proto-Lagrangian. This means the existence of a couple of linear mappings

$\mathbb{K} : \mathbb{C}^{r \times c} \rightarrow \mathbb{C}^{r \times c}, \mathbb{L} : \mathbb{C}^{Nr \times c} \rightarrow \mathbb{C}^{Nr \times c}$ , together with an affine mapping

$\underline{x} \mapsto -s\underline{a} + e^{sA}\underline{x}$ , such that for all matrices  $\mathbb{P} \in \mathbb{C}^{r \times c}, \underline{\mathbb{Q}} \in \mathbb{C}^{Nr \times c}$  and  $\underline{x} \in \mathbb{R}^N$ ,

$$\mathbb{L}(e^{s\mathbb{K}}\mathbb{P}; e^{s\mathbb{L}}\underline{\mathbb{Q}}; -s\underline{a} + e^{sA}\underline{x}) = \mathbb{L}(\mathbb{P}, \underline{\mathbb{Q}}; \underline{x}) + \mathcal{O}(s^2).$$

Of course the presented conservation laws are just special cases of Noether's Theorem.

- For the construction of gauge theories one needs, in physicist's terminology, a 'global symmetry of the Lagrangian'. To achieve this, an *Internal Symmetry* of the proto-Lagrangian  $\mathbb{L}$  is required here: For some fixed Lie-group  $\mathfrak{G} \subset \mathbb{C}^{c \times c}$ , the proto-Lagrangian satisfies

$$\mathbb{L}(\mathbb{P}\mathbb{U}; \underline{\mathbb{Q}}\mathbb{U}; \underline{x}) = \mathbb{L}(\mathbb{P}; \underline{\mathbb{Q}}; \underline{x}), \quad \text{for all } \mathbb{P} \in \mathbb{C}^{r \times c}, \underline{\mathbb{Q}} \in \mathbb{C}^{Nr \times c}, \mathbb{U} \in \mathfrak{G}, \underline{x} \in \mathbb{R}^N.$$

Roughly speaking, a gauge theory for a Lagrangian based system of PDE's is some kind of symmetry preserving extension of the original Lagrangian *density* with new (dependent) 'field'-variables  $\underline{x} \mapsto \underline{A}(\underline{x}) = [\mathcal{A}_1(\underline{x}), \dots, \mathcal{A}_N(\underline{x})]$  on  $\mathbb{R}^N$  added, such that the original 'quantities'  $\underline{\psi}$  become subjected to the 'gauge fields'  $\underline{A}$  and viceversa. Since about a century, Weyl 1918, it is well known that, given the existence of some 'global symmetry group'  $\mathfrak{G}$  of  $\mathbb{L}$ , an extension of type

$$\mathbb{L}_{\psi, A}(\underline{x}) = \mathbb{L}(\underline{\psi}; \nabla \underline{\psi} + \underline{\psi} \cdot \underline{A}; \underline{x}) + \mathbb{G}(\underline{A}; \nabla \underline{A}; \underline{x}),$$

is *often* possible. This extension has to exhibit what physicists call, a '*Local Symmetry*': The Lagrangian density remains unaltered if in  $\mathbb{L}_{\psi, A}$  the quantities  $\underline{\psi}$  and  $\underline{A}$  are, each in their own way, subjected to group actions taken from  $\mathfrak{G}_{\text{loc}} = \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G})$ ,

which is the *group* of smooth maps  $\mathbb{R}^N \rightarrow \mathfrak{G}$ . The added 'gauge fields'  $\underline{A}$  have to take their values in the Lie Algebra  $\mathfrak{g}$  of the symmetry group  $\mathfrak{G}$ .

Summarizing, 'locally symmetric' means, symbolically,

$$\begin{aligned} \mathcal{L}(\underline{\psi}U ; \nabla(\underline{\psi}U) + (\underline{\psi}U) \cdot (\underline{A} \triangleleft U) ; \underline{x}) + \mathcal{G}(\underline{A} \triangleleft U ; \nabla(\underline{A} \triangleleft U) ; \underline{x}) = \\ = \mathcal{L}(\underline{\psi} ; \nabla \underline{\psi} + \underline{\psi} \cdot \underline{A} ; \underline{x}) + \mathcal{G}(\underline{A} ; \nabla \underline{A} ; \underline{x}) , \quad \text{for all } U \in \mathfrak{G}_{\text{loc}} . \end{aligned}$$

- The considerations in the underlying notes not only include the standard hyperbolic evolution equations of pre-quantized fields. Wide classes of parabolic/elliptic systems turn out to have gauge extensions as well. Note the subtle extra condition (5.14) in Thm 5.5 which is, besides internal symmetry of the proto-Lagrangian, necessary for gauge extensions. Its necessity lies in the fact that one has to reconcile the *complex* vector space, in which the  $\underline{\psi}$  take their values, with the *real* vector space  $\mathfrak{g}$ , the Lie-Algebra. In the standard preludes to quantum field the requirement (5.14) is never discussed, but manifestly met with.
- These notes do not contain functional analysis or differential geometry. The reader will find only bare elementary considerations on matrix-valued functions: The columns of the  $\underline{x} \mapsto \underline{\psi}(\underline{x}) \in \mathbb{C}^{r \times c}$  might describe the 'pre-quantized wave functions' of individual elementary particles, whereas the 'components' of  $\underline{x} \mapsto \underline{A}(\underline{x}) \in \mathfrak{g}^N$ , with  $\mathfrak{g} \subset \mathbb{C}^{c \times c}$ , might represent the pre-quantized gauge fields. For an elementary and very readable account on the differential geometrical aspects, see the contributions 3-4 in [JP].

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# 1 Foretaste: Some gauge-type calculations

For functions  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$  we consider, by way of example, the PDE

$$\Gamma^\mu (\partial_\mu \Psi + \Psi \mathcal{A}_\mu) + M \Psi = f, \quad (1.1)$$

with prescribed matrix valued coefficients

$$\Gamma^\mu : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}, \quad \mathcal{A}_\mu : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}, \quad 1 \leq \mu \leq N, \quad M : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r},$$

and prescribed right hand side  $f : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$ . All considered functions are supposed to be sufficiently smooth. The summation convention for upper and lower indices applies.

In physics each column of  $\Psi$  may represent a 'classical-particle wave'. The  $\mathcal{A}_\mu$  may then represent 'gauge fields'.

## Theorem 1.1

Let  $\mathcal{U}, \mathcal{V} : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}$  and suppose them invertible with  $\mathcal{U}^{-1}, \mathcal{V}^{-1} : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}$ . The function  $\hat{\Psi} = \Psi \mathcal{U} : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times k}$ , with  $\Psi$  any solution of (1.1) is a solution of

$$\Gamma^\mu (\partial_\mu \hat{\Psi} + \hat{\Psi} \hat{\mathcal{A}}_\mu) + M \hat{\Psi} = \hat{f}, \quad (1.2)$$

if and only if we take the new coefficients  $\hat{\mathcal{A}}_\mu = \mathcal{U}^{-1} \mathcal{A}_\mu \mathcal{U} - \mathcal{U}^{-1} (\partial_\mu \mathcal{U})$  and  $\hat{f} = f \mathcal{U}$ .

In addition we have  $\hat{\mathcal{A}}_\mu = (\mathcal{U} \mathcal{V})^{-1} \mathcal{A}_\mu (\mathcal{U} \mathcal{V}) - (\mathcal{U} \mathcal{V})^{-1} (\partial_\mu (\mathcal{U} \mathcal{V})) = \mathcal{V}^{-1} \hat{\mathcal{A}}_\mu \mathcal{V} - \mathcal{V}^{-1} (\partial_\mu \mathcal{V})$ .

**Proof:** Multiply (1.1) from the right by  $\mathcal{U}$  and rearrange. ■

In the next Theorem a 'transformation property' for matrix valued functions is derived.

## Theorem 1.2

Let  $\mathcal{A}_\mu : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}$  and  $\hat{\mathcal{A}}_\mu = \mathcal{U}^{-1} \mathcal{A}_\mu \mathcal{U} - \mathcal{U}^{-1} (\partial_\mu \mathcal{U})$ . Define

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - (\mathcal{A}_\mu \mathcal{A}_\nu - \mathcal{A}_\nu \mathcal{A}_\mu). \quad (1.3)$$

Then

$$\hat{\mathcal{F}}_{\mu\nu} = \partial_\mu \hat{\mathcal{A}}_\nu - \partial_\nu \hat{\mathcal{A}}_\mu - (\hat{\mathcal{A}}_\mu \hat{\mathcal{A}}_\nu - \hat{\mathcal{A}}_\nu \hat{\mathcal{A}}_\mu) = \mathcal{U}^{-1} \mathcal{F}_{\mu\nu} \mathcal{U}. \quad (1.4)$$

**Proof:** First note that from  $\partial_\mu (\mathcal{U}^{-1} \mathcal{U}) = \partial_\mu I = 0$  it follows that  $\partial_\mu (\mathcal{U}^{-1}) = -\mathcal{U}^{-1} (\partial_\mu \mathcal{U}) \mathcal{U}^{-1}$ .

Calculate

$$\begin{aligned} \partial_\mu \hat{\mathcal{A}}_\nu &= \partial_\mu (\mathcal{U}^{-1} \mathcal{A}_\nu \mathcal{U} - \mathcal{U}^{-1} (\partial_\nu \mathcal{U})) = \\ &= \mathcal{U}^{-1} (\partial_\mu \mathcal{A}_\nu) \mathcal{U} - \mathcal{U}^{-1} (\partial_\mu \mathcal{U}) \mathcal{U}^{-1} \mathcal{A}_\nu \mathcal{U} + \mathcal{U}^{-1} \mathcal{A}_\nu (\partial_\mu \mathcal{U}) + \mathcal{U}^{-1} (\partial_\mu \mathcal{U}) \mathcal{U}^{-1} (\partial_\nu \mathcal{U}) - \mathcal{U}^{-1} (\partial_\mu \partial_\nu \mathcal{U}). \end{aligned}$$

and

$$\begin{aligned} \hat{\mathcal{A}}_\mu \hat{\mathcal{A}}_\nu &= \{\mathcal{U}^{-1} \mathcal{A}_\mu \mathcal{U} - \mathcal{U}^{-1} (\partial_\mu \mathcal{U})\} \{\mathcal{U}^{-1} \mathcal{A}_\nu \mathcal{U} - \mathcal{U}^{-1} (\partial_\nu \mathcal{U})\} = \\ &= \mathcal{U}^{-1} (\mathcal{A}_\mu \mathcal{A}_\nu) \mathcal{U} - (\mathcal{U}^{-1} \mathcal{A}_\mu \mathcal{U}) (\mathcal{U}^{-1} (\partial_\nu \mathcal{U})) - (\mathcal{U}^{-1} (\partial_\mu \mathcal{U})) (\mathcal{U}^{-1} \mathcal{A}_\nu \mathcal{U}) + (\mathcal{U}^{-1} (\partial_\mu \mathcal{U})) (\mathcal{U}^{-1} (\partial_\nu \mathcal{U})). \end{aligned}$$

Interchange the indices for two more terms and add according to (1.4). All rubbish terms cancel out. ■

We now look for sesqui-linear conservation laws which hold for suitable classes of  $\mathcal{A}_\mu$

### Condition 1.3

$K : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}$ , is such that

i:  $K\Gamma^\mu = (K\Gamma^\mu)^\dagger$ ,   ii:  $\partial_\mu(K\Gamma^\mu) = 0$ ,   iii:  $KM + M^\dagger K^\dagger = 0$ .

Here, the dagger  $\dagger$  denotes 'Hermitean transposition'.

Note that in the important special case that  $\Gamma^\mu = (\Gamma^\mu)^\dagger$ ,  $\Gamma^\mu$  is constant and  $M = -M^\dagger$ , the condition is satisfied by  $K = I$ , the identity matrix. In the case of the Dirac equation one could take  $K = \Gamma^0$ . Cf. [M], Messiah II pp. 890-899. <sup>1</sup>

### Theorem 1.4

Let  $K : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}$  satisfy Condition 1.3.

Fix some  $J \in \mathbb{C}^{c \times c}$ .

Let  $\mathcal{A}_\mu : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}$  satisfy  $\mathcal{A}_\mu^\dagger J + J \mathcal{A}_\mu = 0$ ,  $1 \leq \mu \leq N$ .

Let  $U : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}$  satisfy  $U^\dagger(\underline{x})JU(\underline{x}) = J$ ,  $\underline{x} \in \mathbb{R}^N$ .

a. For any solution  $\Psi$  of (1.1) with  $f = 0$ , there is the conservation law

$$\sum_{\mu=1}^N \partial_\mu \text{Tr}(J^{-1}[\Psi^\dagger K \Gamma^\mu \Psi]) = 0. \quad (1.5)$$

b. This conservation law is a gauge invariant local conservation law.

That means  $\text{Tr}(J^{-1}[\hat{\Psi}^\dagger K \Gamma^\mu \hat{\Psi}]) = \text{Tr}(J^{-1}[\Psi^\dagger K \Gamma^\mu \Psi])$ ,  $1 \leq \mu \leq N$ .

### Proof

a. Take  $f = 0$  in (1.1) and multiply from the left with  $\Psi^\dagger K$ :

$$\Psi^\dagger K \Gamma^\mu (\partial_\mu \Psi) + \Psi^\dagger K \Gamma^\mu \Psi \mathcal{A}_\mu + \Psi^\dagger K M \Psi = 0. \quad (1.6)$$

The Hermitean transpose reads

$$(\partial_\mu \Psi)^\dagger (K \Gamma^\mu)^\dagger \Psi + \mathcal{A}_\mu^\dagger \Psi^\dagger (K \Gamma^\mu)^\dagger \Psi + \Psi^\dagger M^\dagger K^\dagger \Psi = 0. \quad (1.7)$$

Multiply (1.6) from the right with  $J^{-1}$  and (1.7) from the left with  $J^{-1}$ . Add those two identities and take the trace. Use Condition 1.3 and the properties  $\text{Tr}(AB) = \text{Tr}(BA)$ ,  $\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$  and  $\partial_\mu \text{Tr}(A) = \text{Tr}(\partial_\mu A)$ . The sum of the 1st terms of (1.6), (1.7) result in

$$\begin{aligned} & \text{Tr}\{J^{-1}[\Psi^\dagger (K \Gamma^\mu) \partial_\mu \Psi + (\partial_\mu \Psi)^\dagger (K \Gamma^\mu)^\dagger \Psi]\} = \\ & = \partial_\mu \text{Tr}\{J^{-1} \Psi^\dagger (K \Gamma^\mu) \Psi\} - \text{Tr}\{J^{-1} \Psi^\dagger \partial_\mu (K \Gamma^\mu) \Psi\} = \partial_\mu \text{Tr}\{J^{-1} \Psi^\dagger (K \Gamma^\mu) \Psi\}. \end{aligned}$$

The sum of the 2nd terms of (1.6), (1.7) is

$$\text{Tr}\{\Psi^\dagger K \Gamma^\mu \Psi (\mathcal{A}_\mu J^{-1} + J^{-1} \mathcal{A}_\mu^\dagger)\} = 0.$$

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<sup>1</sup>In the non-covariant form, i.e. the original form, of Dirac's equation one has  $\Gamma^0 = I$ ,  $\Gamma^\kappa = \gamma^0 \gamma^\kappa$ ,  $1 \leq \kappa \leq 3$ , where the  $\gamma^\mu$ ,  $0 \leq \mu \leq 3$  are Dirac-Clifford matrices, which make the Dirac equation covariant proof.

The sum of the 3rd terms of (1.6), (1.7)

$$\text{Tr}\{J^{-1}\Psi^\dagger(KM + M^\dagger K^\dagger)\Psi\} = 0.$$

Thus, we find (1.5)

**b.** By putting hats on  $\Psi$  and  $\mathcal{A}_\mu$  our considerations can be rephrased for PDE (1.2). Remind that from  $U^\dagger J U = J$  it follows that  $J^{-1}U^\dagger = U^{-1}J^{-1}$ . Finally

$$\text{Tr}(J^{-1}U^\dagger[\Psi^\dagger K \Gamma^\mu \Psi]U) = \text{Tr}(U^{-1}J^{-1}[\Psi^\dagger K \Gamma^\mu \Psi]U) = \text{Tr}(J^{-1}[\Psi^\dagger K \Gamma^\mu \Psi]).$$

■

## 2 Stationary points of complex-valued functionals

In this section we pay some attention to the Euler Lagrange field equations in the *complex field* case. Most physics textbooks start, in a rather verbose way, with 18th century variational rituals. However most of them become suddenly very vague, or fall completely silent, when state functions involving *complex variables* come into play! In order to get some feeling for such Lagrangians, we first mention a finite dimensional toy result.

### Theorem 2.1

Let

$$f : \mathbb{C}^n \times \mathbb{C}^n \ni (\underline{z}; \underline{w}) \mapsto f(\underline{z}, \underline{w}) \in \mathbb{C}$$

be an analytic function of  $2n$  complex variables with the special property  $f(\underline{z}, \underline{z}^*) \in \mathbb{R}$ , for all  $\underline{z} \in \mathbb{C}^n$ . Here  $\underline{z} = \underline{x} + i\underline{y}$ ,  $\underline{z}^* = \underline{x} - i\underline{y}$ .

**a.** Consider the function

$$\mathbb{R}^n \times \mathbb{R}^n \ni (\underline{x}; \underline{y}) \mapsto g(\underline{x}, \underline{y}) = f(\underline{z}, \underline{z}^*) = f(\underline{x} + i\underline{y}, \underline{x} - i\underline{y}) \in \mathbb{R}.$$

The relations between the (real) partial derivatives of  $g$  at  $(\underline{x}, \underline{y})$  and the (complex) partial derivatives of  $f$  at  $(\underline{z}, \underline{z}^*)$  are

$$\begin{aligned} \frac{\partial g}{\partial \underline{x}}(\underline{x}, \underline{y}) &= \frac{\partial f}{\partial \underline{z}}(z, z^*) + \frac{\partial f}{\partial \underline{w}}(z, z^*) & \frac{\partial f}{\partial \underline{z}}(\underline{z}, \underline{z}^*) &= \frac{1}{2} \left( \frac{\partial g}{\partial \underline{x}}(\underline{x}, \underline{y}) - i \frac{\partial g}{\partial \underline{y}}(\underline{x}, \underline{y}) \right) \\ \frac{\partial g}{\partial \underline{y}}(\underline{x}, \underline{y}) &= i \frac{\partial f}{\partial \underline{z}}(z, z^*) - i \frac{\partial f}{\partial \underline{w}}(z, z^*) & \frac{\partial f}{\partial \underline{w}}(z, z^*) &= \frac{1}{2} \left( \frac{\partial g}{\partial \underline{x}}(\underline{x}, \underline{y}) + i \frac{\partial g}{\partial \underline{y}}(\underline{x}, \underline{y}) \right) \\ & & \frac{\partial f}{\partial \underline{w}}(\underline{z}, \underline{z}^*) &= \overline{\frac{\partial f}{\partial \underline{z}}(\underline{z}, \underline{z}^*)} \end{aligned} \tag{2.1}$$

**b.** For  $g$  to have a stationary point at  $(\underline{a}; \underline{b}) \in \mathbb{R}^n \times \mathbb{R}^n$  **each one** of the following three conditions is necessary and sufficient

$$\begin{aligned}
& \bullet \quad \frac{\partial g}{\partial \underline{x}}(\underline{a}, \underline{b}) = \frac{\partial g}{\partial \underline{y}}(\underline{a}, \underline{b}) = \underline{0}, \\
& \bullet \quad \frac{\partial f}{\partial \underline{z}}(\underline{a} + i\underline{b}, \underline{a} - i\underline{b}) = \underline{0}, \\
& \bullet \quad \frac{\partial f}{\partial \underline{w}}(\underline{a} + i\underline{b}, \underline{a} - i\underline{b}) = " \frac{\partial f}{\partial \underline{z}^*}(\underline{a} + i\underline{b}, \underline{a} - i\underline{b}) " = \underline{0}.
\end{aligned} \tag{2.2}$$

**c.** If the special property  $f(\underline{x} + i\underline{y}, \underline{x} - i\underline{y}) \in \mathbb{R}$  is relaxed to  $\phi(f(\underline{x} + i\underline{y}, \underline{x} - i\underline{y})) \in \mathbb{R}$  for some non-constant analytic  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ , then the 'stationary point result' **b.** still holds.

**Proof:** Straightforward calculation ■

In Theorem 2.4 an  $\infty$ -dimensional generalisation of this result is presented.

### A special bookkeeping

In the sequel, for the above variable  $\underline{z}$ , usually a matrix  $Z \in \mathbb{C}^{r \times c}$  will be taken. In order to explain our bookkeeping and also for some special properties, we now consider an analytic function of 2 matrix variables

$$\mathcal{F} : \mathbb{C}^{r \times c} \times \mathbb{C}^{c \times r} \rightarrow \mathbb{C} : (Z; W) \mapsto \mathcal{F}(Z, W). \tag{2.3}$$

Because of Hartog's Theorem, see [H] Thm 2.2.8, it is enough to assume analyticity with respect to each entry of each matrix separately.

The (complex!) partial derivatives of  $\mathcal{F}$  are gathered in matrices,

$$(Z; W) \mapsto \mathcal{F}^{(1)}(Z, W) \in \mathbb{C}^{c \times r}, \quad (Z; W) \mapsto \mathcal{F}^{(2)}(Z, W) \in \mathbb{C}^{r \times c},$$

with

$$[\mathcal{F}^{(1)}]_{ij} = \left[ \frac{\partial \mathcal{F}}{\partial Z} \right]_{ij} = \frac{\partial \mathcal{F}}{\partial Z_{ji}}, \quad [\mathcal{F}^{(2)}]_{k\ell} = \left[ \frac{\partial \mathcal{F}}{\partial W} \right]_{k\ell} = \frac{\partial \mathcal{F}}{\partial W_{\ell k}}. \tag{2.4}$$

In our notation the  $\mathbb{C}$ -linearization of  $\mathcal{F}$  at  $(Z, W)$ , for  $\varepsilon \in \mathbb{C}$ ,  $|\varepsilon|$  small, reads

$$\mathcal{F}(Z + \varepsilon H, W + \varepsilon K) = \mathcal{F}(Z, W) + \varepsilon \text{Tr}\{[\mathcal{F}^{(1)}]H\} + \varepsilon \text{Tr}\{[\mathcal{F}^{(2)}]K\} + \mathcal{O}(|\varepsilon|^2). \tag{2.5}$$

**Notation:** Sometimes, in order to avoid excessive use of brackets, it is convenient to write  $\text{Tr}\{\mathcal{F}^{(1)} : H\}$  instead of  $\text{Tr}\{[\mathcal{F}^{(1)}]H\}$ .

Also, without warning, in proofs sometimes Einstein's summation convention for repeated upper and lower indices will be used.

Next split  $Z$  in real and imaginary parts  $Z = X + iY$  and introduce the function

$$\widetilde{\mathcal{F}} : \mathbb{R}^{r \times c} \times \mathbb{R}^{r \times c} \rightarrow \mathbb{C} : (X; Y) \mapsto \widetilde{\mathcal{F}}(X, Y) = \mathcal{F}(Z, Z^\dagger) = \mathcal{F}(X + iY, X^\top - iY^\top). \quad (2.6)$$

The  $\mathbb{R}$ -linearization of  $\widetilde{\mathcal{F}}$  at  $(X, Y)$  for  $\varepsilon \in \mathbb{R}$ ,  $|\varepsilon|$  small, can now be written

$$\widetilde{\mathcal{F}}(X + \varepsilon A, Y + \varepsilon B) = \widetilde{\mathcal{F}}(X, Y) + \varepsilon \text{Tr} \left\{ \frac{\partial \widetilde{\mathcal{F}}}{\partial X} A + \frac{\partial \widetilde{\mathcal{F}}}{\partial Y} B \right\} + \mathcal{O}(\varepsilon^2), \quad (2.7)$$

with

$$\begin{aligned} \text{Tr} \left\{ \frac{\partial \widetilde{\mathcal{F}}}{\partial X} A \right\} &= \text{Tr} \{ [\mathcal{F}^{(1)}] A \} + \text{Tr} \{ [\mathcal{F}^{(2)}] A^\top \} = \text{Tr} \{ ([\mathcal{F}^{(1)}] + [\mathcal{F}^{(2)}]^\top) A \}, \\ \text{Tr} \left\{ \frac{\partial \widetilde{\mathcal{F}}}{\partial Y} B \right\} &= \text{Tr} \{ i[\mathcal{F}^{(1)}] B \} + \text{Tr} \{ -i[\mathcal{F}^{(2)}] B^\top \} = \text{Tr} \{ i([\mathcal{F}^{(1)}] - [\mathcal{F}^{(2)}]^\top) B \}, \end{aligned} \quad (2.8)$$

where the matrices  $X, Y, A, B$  are all real. The (complex) derivatives  $\mathcal{F}^{(1)}, \mathcal{F}^{(2)}$  are taken at  $(Z, Z^\dagger)$ . In the usual (somewhat confusing) notation, this corresponds to

$$\frac{\partial \widetilde{\mathcal{F}}}{\partial X} = \frac{\partial \mathcal{F}}{\partial X} = \frac{\partial \mathcal{F}}{\partial Z} + \left[ \frac{\partial \mathcal{F}}{\partial Z^\dagger} \right]^\top, \quad \frac{\partial \widetilde{\mathcal{F}}}{\partial Y} = \frac{\partial \mathcal{F}}{\partial Y} = i \frac{\partial \mathcal{F}}{\partial Z} - i \left[ \frac{\partial \mathcal{F}}{\partial Z^\dagger} \right]^\top, \quad (2.9)$$

and, similarly sloppy,

$$\frac{\partial \mathcal{F}}{\partial Z} = \frac{1}{2} \left( \frac{\partial \mathcal{F}}{\partial X} - i \frac{\partial \mathcal{F}}{\partial Y} \right), \quad \left[ \frac{\partial \mathcal{F}}{\partial Z^\dagger} \right]^\top = \frac{1}{2} \left( \frac{\partial \mathcal{F}}{\partial X} + i \frac{\partial \mathcal{F}}{\partial Y} \right). \quad (2.10)$$

If it happens that  $Z \mapsto \mathcal{F}(Z, Z^\dagger)$  is  $\mathbb{R}$ -valued, the results of Theorem (2.1) can be rephrased.

## Theorem 2.2

Let, as in (2.3),

$$\mathcal{F} : \mathbb{C}^{r \times c} \times \mathbb{C}^{c \times r} \ni (Z; W) \mapsto \mathcal{F}(Z, W) \in \mathbb{C}.$$

be analytic. Suppose  $\mathcal{F}(Z, Z^\dagger) \in \mathbb{R}$ , for all  $Z \in \mathbb{C}^{r \times c}$ . Write  $Z = X + iY$ . Denote

$$\widetilde{\mathcal{F}} : \mathbb{R}^{r \times c} \times \mathbb{R}^{r \times c} \rightarrow \mathbb{R} : (X; Y) \mapsto \widetilde{\mathcal{F}}(X, Y) = \mathcal{F}(Z, Z^\dagger) = \mathcal{F}(X + iY, X^\top - iY^\top),$$

• We have

$$\mathcal{F}^{(1)}(Z, Z^\dagger) = [\mathcal{F}^{(2)}(Z, Z^\dagger)]^\dagger. \quad (2.11)$$

Further, for the function  $\widetilde{\mathcal{F}}$  to have a stationary point at  $(A; B) \in \mathbb{R}^{r \times c} \times \mathbb{R}^{r \times c}$  **each one** of the following three conditions is necessary and sufficient

$$\begin{aligned} &\bullet \quad \frac{\partial \widetilde{\mathcal{F}}}{\partial X}(A, B) = \frac{\partial \widetilde{\mathcal{F}}}{\partial Y}(A, B) = 0 \\ &\bullet \quad \mathcal{F}^{(1)}(A + iB, A^\top - iB^\top) = " \frac{\partial \mathcal{F}}{\partial Z}(A + iB, A^\top - iB^\top) " = 0 \\ &\bullet \quad \mathcal{F}^{(2)}(A + iB, A^\top - iB^\top) = " \frac{\partial \mathcal{F}}{\partial Z^\dagger}(A + iB, A^\top - iB^\top) " = 0. \end{aligned} \quad (2.12)$$



**Proof:** Is mostly a reformulation of the preceding theorem. It follows directly from (2.9)-(2.10). ■

In order to build the concept of **Lagrangian density** we need an analytic function, named **proto-Lagrangian**,

$$\begin{aligned} \mathcal{L} : \mathbb{C}^{r \times c} \times \mathbb{C}^{c \times r} \times \mathbb{C}^{Nr \times c} \times \mathbb{C}^{c \times Nr} \times \mathbb{R}^N &\rightarrow \mathbb{C}, \\ (P; Q^\top; \underline{R}; \underline{S}^\top; \underline{x}) &\mapsto \mathcal{L}(P; Q^\top; \underline{R}; \underline{S}^\top; \underline{x}), \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} P &\in \mathbb{C}^{r \times c}, \quad \underline{R} = \text{col}[R_1, \dots, R_N], \quad R_\mu \in \mathbb{C}^{r \times c}, \quad 1 \leq \mu \leq N, \\ Q^\top &\in \mathbb{C}^{c \times r}, \quad \underline{S}^\top = \text{row}[S_1^\top, \dots, S_N^\top], \quad S_\mu^\top \in \mathbb{C}^{c \times r}, \quad 1 \leq \mu \leq N. \end{aligned}$$

Instead of (2.13) it will be convenient sometimes to denote the proto Lagrangian by

$$\mathcal{L}(P; Q^\top; \dots, \underline{R}_\mu, \dots; \dots, \underline{S}_\mu^\top, \dots; \underline{x}).$$

It will be required that  $\mathcal{L}(O; O^\top; \underline{Q}; \underline{Q}^\top; \underline{x}) = 0$ .

The (complex) partial derivatives of  $\mathcal{L}$ , cf. (2.4)-(2.5), with respect to its  $2N + 2$  matrix arguments are denoted, respectively,

$$\mathcal{L}^{(o)}, \mathcal{L}^{(o\star)}, \mathcal{L}^{(1)}, \dots, \mathcal{L}^{(N)}, \mathcal{L}^{(1\star)}, \dots, \mathcal{L}^{(N\star)}.$$

The (real) partial derivatives of  $\mathcal{L}$ , with respect to the vector variable  $\underline{x}$  is denoted  $\mathcal{L}^{(\nabla)}$ . For any given matrix-valued function  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$ , we define a *Lagrangian density*  $\mathcal{L}_\psi : \mathbb{R}^N \rightarrow \mathbb{C}$ , by substitution of  $\Psi$ , its 1st derivatives  $\partial_\mu \Psi = \Psi_{,\mu}$ ,  $1 \leq \mu \leq N$ , and the hermitean transposed of all those, in  $\mathcal{L}$ :

$$\underline{x} \mapsto \mathcal{L}_\psi(\underline{x}) = \mathcal{L}(\Psi(\underline{x}); \Psi^\dagger(\underline{x}); \nabla \Psi(\underline{x}); \nabla \Psi^\dagger(\underline{x}); \underline{x}), \quad (2.14)$$

where

$$\begin{aligned} \nabla \Psi(\underline{x}) &= \text{col}[\partial_1 \Psi(\underline{x}), \dots, \partial_N \Psi(\underline{x})] \in \mathbb{C}^{Nr \times c}, \\ \nabla \Psi^\dagger(\underline{x}) &= \text{row}[\partial_1 \Psi^\dagger(\underline{x}), \dots, \partial_N \Psi^\dagger(\underline{x})] \in \mathbb{C}^{c \times Nr}. \end{aligned}$$

Also the matrix-valued functions

$$\underline{x} \mapsto [\mathcal{L}_\psi^{(\mu)}](\underline{x}) = [\mathcal{L}^{(\mu)}](\Psi(\underline{x}); \Psi^\dagger(\underline{x}); \nabla \Psi(\underline{x}); \nabla \Psi^\dagger(\underline{x}); \underline{x}) \in \mathbb{C}^{c \times r},$$

similarly  $\underline{x} \mapsto [\mathcal{L}_\psi^{(\mu\star)}] \in \mathbb{C}^{r \times c}$ , and  $\underline{x} \mapsto \mathcal{L}_\psi^{(\nabla)} \in \mathbb{R}^N$ , will be used.

On a suitable space of functions  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$ , it often makes sense to define the **Lagrangian functional**

$$\Psi \mapsto \mathcal{L}(\Psi, \Psi^\dagger) = \int_{\mathbb{R}^N} \mathcal{L}(\Psi(\underline{x}); \Psi^\dagger(\underline{x}); \nabla \Psi(\underline{x}); \nabla \Psi^\dagger(\underline{x}); \underline{x}) d\underline{x} \in \mathbb{C}. \quad (2.15)$$

**Remark 2.3** The Lagrangian functional  $\mathcal{L}$  remains the same if we replace  $\mathcal{L}$  by

$$\mathcal{L}(\Psi; \Psi^\dagger; \nabla \Psi; \nabla \Psi^\dagger; \underline{x}) + \partial_\mu w^\mu(\Psi, \Psi^\dagger, \underline{x}),$$

with  $w^\mu$  a vectorfield which vanishes sufficiently rapidly at infinity.

Therefore the functional  $\Psi \mapsto \mathcal{L}(\Psi, \Psi^\dagger)$  is  $\mathbb{R}$ -valued if

$$\overline{\mathcal{L}(\Psi; \Psi^\dagger; \nabla \Psi; \nabla \Psi^\dagger; \underline{x})} - \mathcal{L}(\Psi; \Psi^\dagger; \nabla \Psi; \nabla \Psi^\dagger; \underline{x}) = \partial_\mu W^\mu(\Psi, \Psi^\dagger, \underline{x}),$$

i.e. the divergence of a vector field.

**Note** that  $\mathcal{L}$  may be  $\mathbb{R}$ -valued while  $\mathcal{L}_\psi$  is not !!

If we split  $\Psi$  into real and imaginary parts:  $\Psi = \Psi_{\text{Re}} + i\Psi_{\text{Im}}$  and  $\Psi_{,\mu} = \Psi_{\text{Re},\mu} + i\Psi_{\text{Im},\mu}$ , the  $\mathbb{R}$ -directional derivatives with respect to  $\Psi_{\text{Re}}$  and  $\Psi_{\text{Im}}$  of the Lagrangian functional  $\mathcal{L}$  are explained by

$$\begin{aligned} \langle \mathcal{D}_{\Psi_{\text{Re}}} \mathcal{L}, \mathbf{A} \rangle &= \frac{d}{d\varepsilon} \mathcal{L}(\Psi + \varepsilon \mathbf{A}, \Psi^\dagger + \varepsilon \mathbf{A}^\top) \Big|_{\varepsilon=0} = \\ &= \frac{d}{d\varepsilon} \int_{\mathbb{R}^N} \mathcal{L}(\Psi(\underline{x}) + \varepsilon \mathbf{A}(\underline{x}); \Psi^\dagger(\underline{x}) + \varepsilon \mathbf{A}^\top(\underline{x}); \nabla(\Psi(\underline{x}) + \varepsilon \mathbf{A}(\underline{x})); \nabla(\Psi^\dagger(\underline{x}) + \varepsilon \mathbf{A}^\top(\underline{x})); \underline{x}) d\underline{x} \Big|_{\varepsilon=0}, \\ &\quad \text{with } \mathbf{A} : \mathbb{R}^N \rightarrow \mathbb{R}^{r \times c}, \text{ and } \varepsilon \in \mathbb{R}, |\varepsilon| \text{ small.} \\ \langle \mathcal{D}_{\Psi_{\text{Im}}} \mathcal{L}, \mathbf{B} \rangle &= \frac{d}{d\varepsilon} \mathcal{L}(\Psi + \varepsilon i\mathbf{B}, \Psi^\dagger - \varepsilon i\mathbf{B}^\top) \Big|_{\varepsilon=0} = \\ &= \frac{d}{d\varepsilon} \int_{\mathbb{R}^N} \mathcal{L}(\Psi(\underline{x}) + \varepsilon i\mathbf{B}(\underline{x}); \Psi^\dagger(\underline{x}) - \varepsilon i\mathbf{B}^\top(\underline{x}); \nabla(\Psi(\underline{x}) + \varepsilon i\mathbf{B}(\underline{x})); \nabla(\Psi^\dagger(\underline{x}) - \varepsilon i\mathbf{B}^\top(\underline{x})); \underline{x}) d\underline{x} \Big|_{\varepsilon=0}, \\ &\quad \text{with } \mathbf{B} : \mathbb{R}^N \rightarrow \mathbb{R}^{r \times c}, \text{ and } \varepsilon \in \mathbb{R}, |\varepsilon| \text{ small.} \end{aligned}$$

When calculating the  $\mathbb{C}$ -directional derivatives  $\mathcal{D}_\Psi \mathcal{L}, \mathcal{D}_{\Psi^\dagger} \mathcal{L}$ , the variables  $\Psi, \Psi^\dagger$  are considered to be independent. These derivatives are supposed to be elements in the (*complex*) *linear* dual of  $\mathbb{L}_2(\mathbb{R}^N; \mathbb{C}^{r \times c})$ . They are explained by

$$\begin{aligned} \langle \mathcal{D}_\Psi \mathcal{L}, \mathbf{H} \rangle &= \frac{d}{d\varepsilon} \mathcal{L}(\Psi + \varepsilon \mathbf{H}, \Psi^\dagger) \Big|_{\varepsilon=0} = \\ &= \frac{d}{d\varepsilon} \int_{\mathbb{R}^N} \mathcal{L}(\Psi(\underline{x}) + \varepsilon \mathbf{H}(\underline{x}); \Psi^\dagger(\underline{x}); \nabla(\Psi(\underline{x}) + \varepsilon \mathbf{H}(\underline{x})); \nabla \Psi^\dagger; \underline{x}) d\underline{x} \Big|_{\varepsilon=0}, \\ &\quad \text{with } \mathbf{H} : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}, \text{ and } \varepsilon \in \mathbb{C}, |\varepsilon| \text{ small.} \\ \langle \mathcal{D}_{\Psi^\dagger} \mathcal{L}, \mathbf{K} \rangle &= \frac{d}{d\varepsilon} \mathcal{L}(\Psi, \Psi^\dagger + \varepsilon \mathbf{K}) \Big|_{\varepsilon=0} = \\ &= \frac{d}{d\varepsilon} \int_{\mathbb{R}^N} \mathcal{L}(\Psi(\underline{x}); \Psi^\dagger(\underline{x}) + \varepsilon \mathbf{K}(\underline{x}); \nabla \Psi(\underline{x}); \nabla(\Psi^\dagger(\underline{x}) + \varepsilon \mathbf{K}(\underline{x})); \underline{x}) d\underline{x} \Big|_{\varepsilon=0}, \\ &\quad \text{with } \mathbf{K} : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times r}, \text{ and } \varepsilon \in \mathbb{C}, |\varepsilon| \text{ small.} \end{aligned}$$

For  $\mathbf{H}, \mathbf{K}, \mathbf{A}, \mathbf{B}$  vanishing sufficiently rapidly at  $\infty$  a partial integration leads to the standard Euler-Lagrange expressions for the functional derivatives of  $\mathcal{L}$ .

**Theorem 2.4**

Assume that  $\mathcal{L}$  is  $\mathbb{R}$ -valued. (Cf. Remark 2.3). If  $\Psi$  satisfies any one of the following three Lagrangian systems

$$\begin{aligned} \mathcal{D}_{\Psi} \mathcal{L} &= [\mathcal{L}_{\psi}^{(o)}] - \sum_{\mu=1}^N \frac{\partial}{\partial x^{\mu}} [\mathcal{L}_{\psi}^{(\mu)}] = 0, \\ \mathcal{D}_{\Psi^{\dagger}} \mathcal{L} &= [\mathcal{L}_{\psi}^{(o\star)}] - \sum_{\mu=1}^N \frac{\partial}{\partial x^{\mu}} [\mathcal{L}_{\psi}^{(\mu\star)}] = 0, \end{aligned} \quad \left\{ \begin{aligned} \mathcal{D}_{\Psi_{\text{Re}}} \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Re}}} - \sum_{\mu=1}^N \frac{\partial}{\partial x^{\mu}} \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Re},\mu}} = 0, \\ \mathcal{D}_{\Psi_{\text{Im}}} \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Im}}} - \sum_{\mu=1}^N \frac{\partial}{\partial x^{\mu}} \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Im},\mu}} = 0. \end{aligned} \right. , \quad (2.16)$$

with  $\mathcal{L} = \mathcal{L}(\Psi(\underline{x}); \Psi^{\dagger}(\underline{x}); \nabla \Psi(\underline{x}); \nabla \Psi^{\dagger}(\underline{x}); \underline{x})$ , then it also satisfies the other two.

**Proof:** With the notation (2.8)-(2.10) we obtain

$$\frac{\partial \mathcal{L}}{\partial \Psi_{\text{Re}}} = \mathcal{L}^{(o)} + [\mathcal{L}^{(o\star)}]^{\top}, \quad \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Im}}} = i\mathcal{L}^{(o)} - i[\mathcal{L}^{(o\star)}]^{\top}, \quad (2.17)$$

and, the other way round,

$$[\mathcal{L}^{(o\star)}]^{\top} = \frac{1}{2} \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Re}}} + i \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Im}}} \right), \quad \mathcal{L}^{(o)} = \frac{1}{2} \left( \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Re}}} - i \frac{\partial \mathcal{L}}{\partial \Psi_{\text{Im}}} \right), \quad (2.18)$$

and similar expressions with  $(o)$ ,  $(o\star)$  replaced by  $(\mu)$ ,  $(\mu\star)$  and  $\Psi$ ,  $\Psi_{\text{Re}}$ ,  $\Psi_{\text{Im}}$  replaced by  $\Psi_{,\mu}$ ,  $\Psi_{\text{Re},\mu}$ ,  $\Psi_{\text{Im},\mu}$ . Then

$$\begin{aligned} \mathcal{D}_{\Psi} \mathcal{L} &= \frac{1}{2} (\mathcal{D}_{\Psi_{\text{Re}}} \mathcal{L} - i \mathcal{D}_{\Psi_{\text{Im}}} \mathcal{L}) & \mathcal{D}_{\Psi_{\text{Re}}} \mathcal{L} &= \mathcal{D}_{\Psi} \mathcal{L} + [\mathcal{D}_{\Psi^{\dagger}} \mathcal{L}]^{\top} \\ [\mathcal{D}_{\Psi^{\dagger}} \mathcal{L}]^{\top} &= \frac{1}{2} (\mathcal{D}_{\Psi_{\text{Re}}} \mathcal{L} + i \mathcal{D}_{\Psi_{\text{Im}}} \mathcal{L}) & [\mathcal{D}_{\Psi_{\text{Im}}} \mathcal{L}]^{\top} &= i \mathcal{D}_{\Psi} \mathcal{L} - i [\mathcal{D}_{\Psi^{\dagger}} \mathcal{L}]^{\top} \end{aligned} .$$

If we take into account that the entries of the matrix valued functions  $\mathcal{D}_{\Psi_{\text{Re}}} \mathcal{L}$  and  $\mathcal{D}_{\Psi_{\text{Im}}} \mathcal{L}$  are  $\mathbb{R}$ -valued, we find

$$[\mathcal{D}_{\Psi^{\dagger}} \mathcal{L}]^{\dagger} = [\mathcal{D}_{\Psi} \mathcal{L}], \quad (2.19)$$

from which the theorem easily follows. ■

**Examples 2.5 (Matter Fields)**

**a)** Let  $\Gamma^{\mu}$  and  $M$  be constant complex matrices with  $\Gamma^{\mu\dagger} = \Gamma^{\mu}$  and  $M = -M^{\dagger}$ . Then the Lagrangian density

$$\mathcal{L}_{\psi} = i \text{Tr} \{ \Psi^{\dagger} \Gamma^{\mu} \partial_{\mu} \Psi + \Psi^{\dagger} M \Psi \}, \quad (2.20)$$

for  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$ , satisfies the condition of Theorem (2.4) and leads to (1.1) with  $\mathcal{A} = 0$ .

**b)** Let  $\Gamma_{\mu}, 1 \leq \mu \leq N : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}$ . Let  $\mathcal{A}_{\mu}, 1 \leq \mu \leq N : \mathbb{R}^N \rightarrow \mathbb{C}^{c \times c}$ .

Let  $M : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}$ .

Suppose both the existence of  $K : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}$ , having inverse  $K^{-1}(\underline{x})$ , for all  $\underline{x} \in \mathbb{R}^N$ , and an invertible  $J \in \mathbb{C}^{c \times c}$  with  $J^{\dagger} = J$ , such that:

$(K\Gamma^\mu)^\dagger = K\Gamma^\mu$ , ,  $1 \leq \mu \leq N$ ,  $\mathcal{A}_\mu^\dagger(\underline{x})J + J\mathcal{A}_\mu(\underline{x}) = 0$ ,  $1 \leq \mu \leq N$ ,  $\underline{x} \in \mathbb{R}^N$ ,  
and  $KM + M^\dagger K^\dagger - \partial_\mu(K\Gamma^\mu) = 0$ .

Then the Lagrangian density

$$\mathcal{L}_\psi = i \text{Tr} \{ \Psi^\dagger K (\Gamma^\mu \partial_\mu \Psi) J^{-1} + \Psi^\dagger K (\Gamma^\mu \Psi \mathcal{A}_\mu) J^{-1} + \Psi^\dagger K M \Psi J^{-1} \}, \quad (2.21)$$

for  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$  satisfies  $\mathcal{L} - \overline{\mathcal{L}} = \partial_\mu w$  and hence the condition of Theorem (2.4).  
It leads to the 'matter-field equation'

$$\Gamma^\mu \partial_\mu \Psi + \Gamma^\mu \Psi \mathcal{A}_\mu + M \Psi = 0 \quad (2.22)$$

**Indeed.** Taking suitable combinations we find respectively

$$\begin{aligned} \text{Tr} \{ \Psi^\dagger K \Gamma^\mu (\partial_\mu \Psi) J^{-1} + J^{-1} (\partial_\mu \Psi)^\dagger (K \Gamma^\mu)^\dagger \Psi \} &= \text{Tr} \{ J^{-1} \partial_\mu [\Psi^\dagger K \Gamma^\mu \Psi] \} + \\ &\quad \text{Tr} \{ J^{-1} [\Psi^\dagger \partial_\mu (K \Gamma^\mu) \Psi] \}, \\ \text{Tr} \{ \Psi^\dagger K (\Gamma^\mu \Psi \mathcal{A}_\mu) J^{-1} + J^{-1} \mathcal{A}_\mu^\dagger \Psi^\dagger (K \Gamma^\mu)^\dagger \Psi \} &= \text{Tr} \{ [\mathcal{A}_\mu J^{-1} + J^{-1} \mathcal{A}_\mu^\dagger] \Psi^\dagger (K \Gamma^\mu) \Psi \} = 0, \\ \text{Tr} \{ \Psi^\dagger K M \Psi J^{-1} + J^{-1} \Psi^\dagger M^\dagger K^\dagger \Psi \} &= \text{Tr} \{ J^{-1} \Psi^\dagger K M \Psi + J^{-1} \Psi^\dagger M^\dagger K^\dagger \Psi \} = \\ &= \text{Tr} \{ J^{-1} [\Psi^\dagger (K M + M^\dagger K^\dagger) \Psi] \}. \end{aligned}$$

Ultimately we find

$$\mathcal{L}_\psi - \overline{\mathcal{L}_\psi} = \partial_\mu \text{Tr} \{ J^{-1} [\Psi^\dagger K \Gamma^\mu \Psi] \} = \partial_\mu \text{Tr} \{ [\Psi^\dagger K \Gamma^\mu \Psi] J^{-1} \}. \quad (2.23)$$

The Euler-Lagrange equations are

$$K (\Gamma^\mu \partial_\mu \Psi + \Gamma^\mu \Psi \mathcal{A}_\mu + M \Psi) J^{-1} = 0, \quad (2.24)$$

from which  $K$  and  $J^{-1}$  can be cancelled.

c) The Lagrangian density

$$\mathcal{L}_\psi = \text{Tr} \{ [\partial_\mu \Psi]^\dagger \Theta^{\mu\nu} [\partial_\nu \Psi] + \Psi^\dagger R \Psi \}, \quad (2.25)$$

with  $\Theta^{\mu\nu}, R : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times r}$  and  $[\Theta^{\mu\nu}]^\dagger = \Theta^{\nu\mu}$ ,  $R^\dagger = R$ , is  $\mathbb{R}$ -valued. It leads to the 2nd order equation

$$\sum_{\mu, \nu} \frac{\partial}{\partial x^\mu} \Theta^{\mu\nu} \frac{\partial}{\partial x^\nu} \Psi - R \Psi = 0. \quad (2.26)$$

d. The Lagrangian density for functions  $\Psi = \text{col} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} : \mathbb{R}^{N+1} \rightarrow \mathbb{C}^2$ ,

$$\mathcal{L}_\psi = \text{Tr} \left[ \Psi^\dagger (i \partial_t \Psi + \Delta \Psi + V \Psi) \right], \quad \text{with } \underline{x} \mapsto V(\underline{x}) \in \mathbb{C}^{2 \times 2}, \quad V^\dagger = V, \quad (2.27)$$

leads to a  $\mathbb{R}$ -valued Lagrangian functional  $\mathcal{L}$ . Indeed

$$\mathcal{L}_\psi - \overline{\mathcal{L}_\psi} = i \partial_t \text{Tr} [\Psi^\dagger \Psi] + \partial_{x_1} \text{Tr} [\Psi^\dagger (\partial_{x_1} \Psi - (\partial_{x_1} \Psi)^\dagger \Psi)] + \dots + \partial_{x_N} \text{Tr} [\Psi^\dagger (\partial_{x_N} \Psi - (\partial_{x_N} \Psi)^\dagger \Psi)].$$

The  $\mathcal{L}_\psi$  of (2.27) leads to the Schrödinger equation for a particle with spin  $\frac{1}{2}$ .

### 3 Free Gauge Fields

The 'field variables' to be considered in this section are smooth functions

$$\underline{\mathcal{A}} : \mathbb{R}^N \rightarrow \underbrace{\mathbb{C}^{c \times c} \times \cdots \times \mathbb{C}^{c \times c}}_{N \text{ times}} : \underline{x} \mapsto \underline{\mathcal{A}}(\underline{x}) = \text{col}[\mathcal{A}_1(\underline{x}), \dots, \mathcal{A}_\mu(\underline{x}), \dots, \mathcal{A}_N(\underline{x})], \quad (3.1)$$

with  $\mathcal{A}_\mu(\underline{x}) \in \mathfrak{g}$ , with  $\mathfrak{g} \subset \mathbb{C}^{c \times c}$  some fixed **real** Lie algebra.<sup>2</sup> This means that  $\mathfrak{g}$  is a  $\mathbb{R}$ -linear subspace in  $\mathbb{C}^{c \times c}$  which is not necessarily a  $\mathbb{C}$ -linear subspace. On  $\mathfrak{g}$  we impose the usual 'commutator'-Lie product

$$\{A_\mu, A_\nu\} = (A_\mu A_\nu - A_\nu A_\mu).$$

Important examples are matrix Lie Algebras of type

$$\mathfrak{g}_J = \{X \in \mathbb{C}^{r \times r} \mid X^\dagger J + JX = 0\}, \quad \text{with fixed invertible } J \in \mathbb{C}^{r \times r}.$$

Note that  $\mathfrak{g}_J$  is always a  $\mathbb{R}$ -linear subspace in  $\mathbb{C}^{r \times r}$ , but not necessarily  $\mathbb{C}$ -linear.

However:  $\{J^{-1} = J^\dagger\} \Rightarrow \{X \in \mathfrak{g}_J \Rightarrow X^\dagger \in \mathfrak{g}_J\}$ .

Next, by  $\mathcal{P}_{\mathfrak{g}} : \mathbb{C}^{c \times c} \rightarrow \mathfrak{g}$ , we denote the **real** orthogonal projection with respect to the **real** inner product  $X, Y \mapsto \text{Re Tr}[X^\dagger Y]$ .

#### Remarks 3.1

Consider  $\mathbb{C}^{c \times c}$  as a **real** vector space with standard **real** inner product  $X, Y \mapsto \text{Re Tr}[X^\dagger Y]$ . By  $\mathcal{P}_{\mathfrak{g}} : \mathbb{C}^{c \times c} \rightarrow \mathfrak{g}$ , we denote the **real** orthogonal projection with respect this inner product.

- The Hermitean conjugation map  $X \mapsto X^\dagger$  is  $\mathbb{R}$ -linear symmetric and orthogonal.
- If  $\forall X \in \mathfrak{g} : X^\dagger \in \mathfrak{g}$ , in short  $\mathfrak{g}^\dagger = \mathfrak{g}$ , it follows that  $\forall X \in \mathbb{C}^{c \times c} : \mathcal{P}_{\mathfrak{g}}(X^\dagger) = (\mathcal{P}_{\mathfrak{g}}X)^\dagger$ .
- For fixed  $K, L \in \mathbb{C}^{c \times c}$  the mapping  $X \mapsto KX^\dagger L$  is  $\mathbb{R}$ -linear. Its  $\mathbb{R}$ -adjoint is  $Y \mapsto LY^\dagger K$ .
- For any fixed invertible  $J \in \mathbb{C}^{c \times c}$  the mapping

$$\mathcal{Q}_J : \mathbb{C}^{c \times c} \rightarrow \mathbb{C}^{c \times c} : X \mapsto \mathcal{Q}_J X = \frac{1}{2}(X - J^{-1}X^\dagger J), \quad (3.2)$$

is a  $\mathbb{R}$ -linear mapping which reduces to the identity map when restricted to  $\mathfrak{g}_J$ .

- $\mathcal{Q}_J$  is a  $\mathbb{R}$ -linear projection on  $\mathfrak{g}_J$  iff  $J = J^\dagger$ .
- $\mathcal{Q}_J$  is a  $\mathbb{R}$ -linear orthogonal projection on  $\mathfrak{g}_J$  if  $J = J^{-1} = J^\dagger$ .  
In this special case  $\mathcal{Q}_J = \mathcal{P}_{\mathfrak{g}}$ , with  $\mathfrak{g} = \mathfrak{g}_J$ .

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<sup>2</sup>In physics textbooks one often denotes  $iA_\mu$ , instead of  $A_\mu$ , cf. [DM]. For resemblance with Electromagnetism, I suppose. Because of  $u(1) = i\mathbb{R}$  ? To this author the factor  $i$  is not convenient in all other cases.

- If we modify the standard real inner product on  $\mathbb{C}^{c \times c}$  to  $X, Y \mapsto \text{Re Tr}[X^\dagger J^2 Y]$ , the projection  $\mathcal{Q}_J$  is orthogonal iff  $J = J^\dagger$ .

**Proof**

- $\text{Re Tr}[(X^\dagger)^\dagger Y] = \text{Re Tr}[XY] = \text{Re Tr}[X^\dagger(Y^\dagger)]$ . Also  $\text{Re Tr}[(X^\dagger)^\dagger(Y^\dagger)] = \text{Re Tr}[(X)^\dagger(Y)]$ .
- Since  $\mathfrak{g}$  is supposed to be an invariant subspace for  $X \mapsto X^\dagger$  and the latter is symmetric, also  $\mathfrak{g}^\perp$  is invariant.
- $\text{Re Tr}[(KX^\dagger L)^\dagger Y] = \text{Re Tr}[KX^\dagger LY^\dagger] = \text{Re Tr}[X^\dagger(LY^\dagger K)]$ .
- For  $X \in \mathfrak{g}$  holds  $(I - \mathcal{Q}_J)X = 0$ , iff  $X \in \mathfrak{g}$ .
- $Q_J^2 = Q_J$  iff  $J = J^\dagger$ .
- $\frac{1}{2} \text{Re Tr}[(X - J^{-1}X^\dagger J)^\dagger J^2 Y] = \frac{1}{2} \text{Re Tr}[X^\dagger J^2 Y] - \frac{1}{2} \text{Re Tr}[X^\dagger J^2 (J^{-1}Y^\dagger J^{\dagger 2} J^{-1})]$ .  
The 2nd term equals  $-\frac{1}{2} \text{Re Tr}[X^\dagger J^2 (J^{-1}Y^\dagger J)]$ , for all  $X, Y$ , iff  $J = J^\dagger$ . ■

Associated with  $\underline{A}$ , cf. (3.1), we introduce covariant-type partial derivatives  $\nabla_\mu^A, 1 \leq \mu \leq N$  of functions  $U \in \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}^{c \times c})$  by

$$\nabla_\mu^A U = \partial_\mu U - \{\mathcal{A}_\mu, U\} = \partial_\mu U - \text{ad}_{\mathcal{A}_\mu} U. \quad (3.3)$$

One has the Leibniz-type rules

$$\begin{aligned} \nabla_\mu^A(UV) &= (\nabla_\mu^A U)V + U(\nabla_\mu^A V), \\ \text{Tr}[U(\nabla_\mu^A V)] &= \partial_\mu \text{Tr}[UV] - \text{Tr}[(\nabla_\mu^A U)V]. \end{aligned} \quad (3.4)$$

Note that if  $U \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g})$  then also  $\nabla_\mu^A U \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g})$ .

Next, as in section 1, for given  $\mathcal{A}_\mu, \mathcal{A}_\nu \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g}), 1 \leq \mu, \nu \leq N$ , define

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \{\mathcal{A}_\mu, \mathcal{A}_\nu\} \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g}), \quad (3.5)$$

to which Theorem 1.2 applies.

For the construction of a  $\mathbb{R}$ -valued Lagrangian density  $\mathcal{G}_A$  for the Gauge field(s)  $\underline{A}$  we again employ a proto Lagrangian  $\mathcal{G}$ , which is now an analytic function of  $N(N-1)$  complex-matrix variables and just smooth in  $N$  real variables:

$$\mathcal{G} : \underbrace{\mathbb{C}^{c \times c} \times \dots \times \mathbb{C}^{c \times c}}_{\frac{1}{2}N(N-1) \text{ times}} \times \underbrace{\mathbb{C}^{c \times c} \times \dots \times \mathbb{C}^{c \times c}}_{\frac{1}{2}N(N-1) \text{ times}} \times \mathbb{R}^N \rightarrow \mathbb{C}. \quad (3.6)$$

The 1st set of entries to this function is labeled by the ordered pairs  $(\mu\nu), 1 \leq \mu < \nu \leq N$ . The 2nd set of entries is labelled by the ordered triple  $(\theta\rho\star), 1 \leq \theta < \rho \leq N$ . We denote

$$\{\dots, P_{\mu\nu}, \dots; \dots, Q_{\theta\rho\star}, \dots; \underline{x}\} \mapsto \mathcal{G}(\dots P_{\mu\nu}, \dots; \dots Q_{\theta\rho\star}, \dots; \underline{x}) \in \mathbb{C},$$

with  $1 \leq \mu < \nu \leq N$  and  $1 \leq \theta < \rho \leq N$ . The 3 bunches of variables get their corresponding partial derivatives denoted by, respectively, cf. (2.4),

$$\mathcal{G}^{(\mu\nu)}(\dots, P_{\theta\rho}, \dots; \dots, Q_{\theta\rho\star}, \dots; \underline{x}), \quad \mathcal{G}^{(\theta\rho\star)}(\dots, P_{\theta\rho}, \dots; \dots, Q_{\theta\rho\star}, \dots; \underline{x}), \quad \mathcal{G}^{(\nabla)}.$$

Let the Lie algebra  $\mathfrak{g}$  be fixed. On  $\mathcal{G}$  we put the condition, take  $Q_{\theta\rho\star} = P_{\theta\rho}^\dagger$ ,

$$\forall \{P_{\mu\nu}\}_{1 \leq \mu < \nu \leq N} \subset \mathfrak{g} \quad \forall \underline{x} \in \mathbb{R}^N : \mathcal{G}(\dots, P_{\mu\nu}, \dots; \dots, P_{\theta\rho}^\dagger, \dots; \underline{x}) \in \mathbb{R}. \quad (3.7)$$

The Lagrangian density we want to consider is found by replacing  $P_{\mu\nu} \rightarrow \mathcal{F}_{\mu\nu}$ ,  $Q_{\theta\rho\star} \rightarrow \mathcal{F}_{\theta\rho}^\dagger$ ,

$$\underline{x} \mapsto \mathcal{G}_A(\underline{x}) = \mathcal{G}(\dots, \mathcal{F}_{\mu\nu}(\underline{x}), \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger(\underline{x}), \dots; \underline{x}) \in \mathbb{R}. \quad (3.8)$$

Note that if  $\mathfrak{g} = \mathfrak{g}_J$ , for some fixed  $J \in \mathbb{C}^{c \times c}$ , we have  $\mathcal{F}_{\theta\rho}^\dagger = -J\mathcal{F}_{\theta\rho}J^{-1}$ ,  $\theta < \rho$ .

As in the previous section, a corresponding useful notation is

$$\underline{x} \mapsto \mathcal{G}_A^{(\mu\nu)}(\underline{x}) = \mathcal{G}^{(\mu\nu)}(\dots, \mathcal{F}_{\mu\nu}(\underline{x}), \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger(\underline{x}), \dots; \underline{x}) \in \mathbb{C}^{c \times c}. \quad (3.9)$$

The Lagrangian density  $\mathcal{G}_A$  depends on the field variables  $\underline{x} \mapsto \mathcal{A}_\mu(\underline{x})$ ,  $1 \leq \mu \leq N$ , and their derivatives. All being functions in a vectorspace over  $\mathbb{R}$ . In the important special case  $\mathfrak{g} = \mathfrak{g}_J$  the hermitean conjugate notation of the field variables  $\mathcal{A}_\mu$  need not even occur. Finally, note that, because of (2.11) and (3.8), we have

$$\mathcal{G}_A^{(\theta\rho\star)}(\underline{x}) = (\mathcal{G}_A^{(\theta\rho)})^\dagger(\underline{x}), \quad 1 \leq \theta < \rho \leq N. \quad (3.10)$$

**Notation 3.2** In order to visually simplify the formulae to come, it is useful to extend the set of functions  $\mathcal{G}_A^{(\mu\nu)}$ , cf.(3.9), to 'full' labels  $1 \leq \mu, \nu \leq N$  in the following way,

$$\hat{\mathcal{G}}_A^{(\mu\nu)} = \begin{cases} \mathcal{G}_A^{(\mu\nu)} & \text{if } 0 \leq \mu < \nu \leq N, \text{ as before,} \\ 0 & \text{if } \mu = \nu, \\ -\mathcal{G}_A^{(\nu\mu)} & \text{if } 0 \leq \nu < \mu \leq N. \end{cases} \quad (3.11)$$

### Theorem 3.3

Fix a matrix Lie algebra  $\mathfrak{g} \subset \mathbb{C}^{c \times c}$ . Consider the Lagrangian density  $\mathcal{G}_A$  of (3.8).

**A.** The Euler-Lagrange equations for the free gauge fields  $\mathcal{A}_\mu$ ,  $1 \leq \mu \leq N$ , with values in the Lie algebra  $\mathfrak{g} \subset \mathbb{C}^{c \times c}$ , read

$$\sum_{\mu=1}^N \mathcal{P}_{\mathfrak{g}} \left( \left( \nabla_\mu^A ([\mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa\star)}]^\dagger) \right)^\dagger \right) = 0, \quad 1 \leq \kappa \leq N, \quad (3.12)$$

with  $\nabla_\mu^A$  as in (3.3).

**B.** In the special case  $\mathfrak{g}^\dagger = \mathfrak{g}$  the Euler-Lagrange equations simplify to

$$\sum_{\mu=1}^N \left( \nabla_\mu^A \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} \right) = 0, \quad 1 \leq \kappa \leq N. \quad (3.13)$$

**C.** If we take  $\mathfrak{g} = \mathfrak{g}_J$ , with  $J = J^\dagger = J^{-1}$ , the latter becomes

$$\sum_{\mu=1}^N \nabla_\mu^A \left( \mathcal{Q}_J [\hat{\mathcal{G}}_A^{(\mu\kappa)}] \right) = 0, \quad 1 \leq \kappa \leq N, \quad (3.14)$$

where  $\mathcal{Q}_J Z = \frac{1}{2}Z - \frac{1}{2}JZ^\dagger J$ ,  $Z \in \mathbb{C}^{c \times c}$ .

**Proof**

**A.** In order to calculate the (directional) derivatives of the Lagrangian functional  $\mathcal{G} = \int \mathcal{G}_A \, d\underline{x}$  with respect to the free gauge fields  $\mathcal{A}_\kappa, 1 \leq \kappa \leq N$ , we first expand a perturbation of  $\underline{x} \mapsto \mathcal{F}_{\mu\nu}(\underline{x})$  by substitution of the gauge fields  $\underline{x} \mapsto \mathcal{A}_\mu(\underline{x}) + \varepsilon \delta_{\mu\kappa} \mathcal{H}(\underline{x}), \varepsilon \in \mathbb{R}$ ,

$$\begin{aligned} \mathcal{F}_{\mu\nu;\varepsilon,\kappa} &= \left[ \partial_\mu (\mathcal{A}_\nu + \varepsilon \delta_{\nu\kappa} \mathcal{H}) - \partial_\nu (\mathcal{A}_\mu + \varepsilon \delta_{\mu\kappa} \mathcal{H}) - \{\mathcal{A}_\mu + \varepsilon \delta_{\mu\kappa} \mathcal{H}, \mathcal{A}_\nu + \varepsilon \delta_{\nu\kappa} \mathcal{H}\} \right] = \\ &= \left[ \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \{\mathcal{A}_\mu, \mathcal{A}_\nu\} \right] + \varepsilon \delta_{\nu\kappa} \left[ \partial_\mu \mathcal{H} - \{\mathcal{A}_\mu, \mathcal{H}\} \right] - \varepsilon \delta_{\mu\kappa} \left[ \partial_\nu \mathcal{H} - \{\mathcal{A}_\nu, \mathcal{H}\} \right] = \\ &= \mathcal{F}_{\mu\nu} + \varepsilon \delta_{\nu\kappa} \nabla_\mu^A \mathcal{H} - \varepsilon \delta_{\mu\kappa} \nabla_\nu^A \mathcal{H}. \end{aligned}$$

Consider the expansion

$$\begin{aligned} &\mathcal{G}(\dots, \mathcal{F}_{\mu\nu;\varepsilon,\kappa}, \dots; \dots, \mathcal{F}_{\theta\rho;\varepsilon,\kappa}^\dagger, \dots; \underline{x}) - \mathcal{G}(\dots, \mathcal{F}_{\mu\nu}, \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger, \dots; \underline{x}) = \\ &= \varepsilon \sum_{1 \leq \mu < \nu \leq N} \text{Tr} \left[ [\mathcal{G}_A^{(\mu\nu)}] [\delta_{\nu\kappa} \nabla_\mu^A \mathcal{H} - \delta_{\mu\kappa} \nabla_\nu^A \mathcal{H}] + \right. \\ &\quad \left. + \varepsilon \sum_{1 \leq \theta < \rho \leq N} \text{Tr} \left[ [\mathcal{G}_A^{(\theta\rho\star)}] [\delta_{\rho\kappa} \nabla_\theta^A \mathcal{H} - \delta_{\theta\kappa} \nabla_\rho^A \mathcal{H}_\kappa]^\dagger \right] + \mathcal{O}(\varepsilon^2) = \right. \\ &= \frac{\varepsilon}{2} \sum_{\mu, \nu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\nu)}] [\delta_{\nu\kappa} \nabla_\mu^A \mathcal{H} - \delta_{\mu\kappa} \nabla_\nu^A \mathcal{H}] + \right. \\ &\quad \left. + \frac{\varepsilon}{2} \sum_{\theta, \rho=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\theta\rho\star)}] [\delta_{\rho\kappa} \nabla_\theta^A \mathcal{H} - \delta_{\theta\kappa} \nabla_\rho^A \mathcal{H}]^\dagger \right] + \mathcal{O}(\varepsilon^2) = \right. \\ &= \frac{\varepsilon}{2} \sum_{\mu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\kappa)}] [\nabla_\mu^A \mathcal{H}] \right] - \frac{\varepsilon}{2} \sum_{\nu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\kappa\nu)}] [\nabla_\nu^A \mathcal{H}] \right] + \\ &\quad + \frac{\varepsilon}{2} \sum_{\theta=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\theta\kappa\star)}] [\nabla_\theta^A \mathcal{H}] \right] - \frac{\varepsilon}{2} \sum_{\rho=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\kappa\rho\star)}] [\nabla_\rho^A \mathcal{H}]^\dagger \right] + \mathcal{O}(\varepsilon^2) = \\ &= \varepsilon \sum_{\mu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\kappa)}] [\nabla_\mu^A \mathcal{H}] \right] + \varepsilon \sum_{\mu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\kappa\star)}] [\nabla_\mu^A \mathcal{H}]^\dagger \right] + \mathcal{O}(\varepsilon^2) = \\ &= 2\varepsilon \text{Re} \sum_{\mu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\kappa\star)}]^\dagger [\nabla_\mu^A \mathcal{H}] \right] + \mathcal{O}(\varepsilon^2) = 2\varepsilon \text{Re} \sum_{\mu=1}^N \text{Tr} \left[ [\mathcal{P}_\mathfrak{g} \hat{\mathcal{G}}_A^{(\mu\kappa\star)}]^\dagger [\nabla_\mu^A \mathcal{H}] \right] + \mathcal{O}(\varepsilon^2) = \\ &= -2\varepsilon \text{Re} \sum_{\mu=1}^N \text{Tr} \left[ \nabla_\mu^A ([\mathcal{P}_\mathfrak{g} \hat{\mathcal{G}}_A^{(\mu\kappa\star)}]^\dagger) \mathcal{H} \right] + \sum_{\mu=1}^N \partial_\mu(\dots) + \mathcal{O}(\varepsilon^2) = \end{aligned}$$



$$= -2\varepsilon \text{Re} \sum_{\mu=1}^N \text{Tr} \left[ \left( \mathcal{P}_{\mathfrak{g}} \left( \left( \nabla_{\mu}^A ([\mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa\star)}]^\dagger) \right)^\dagger \right)^\dagger \mathcal{H} \right] + \sum_{\mu=1}^N \partial_{\mu}(\dots) + \mathcal{O}(\varepsilon^2). \quad (3.15)$$

In this derivation we used, respectively, the antisymmetry  $\mu \leftrightarrow \nu$  of  $[\hat{\mathcal{G}}_A^{(\mu\nu)}]$  and  $[\delta_{\nu\kappa} \nabla_{\mu}^A \mathcal{H} - \delta_{\mu\kappa} \nabla_{\nu}^A \mathcal{H}]$ , the Leibniz rule(3.4), the fact that  $\text{Re Tr}[(\dots)^\dagger \mathcal{H}]$  expresses the real inner product on  $\mathbb{C}^{c \times c}$  and  $\mathcal{P}_{\mathfrak{g}}$  the real orthogonal projection on  $\mathfrak{g}$ .

Also properties like  $\text{Tr}[AB] = \text{Tr}[BA]$ ,  $\text{Tr}[A\{B, C\}] = \text{Tr}[\{A, B\}C]$  play a crucial role. The result now follows by the usual variational practices.

**B.** If  $\mathfrak{g}^\dagger = \mathfrak{g}$  the real linear mappings  $\{.\}^\dagger$  and  $\mathcal{P}_{\mathfrak{g}}$  commute, which greatly simplifies the result of A.

**C.** Use Remarks 3.1. ■

### Example 3.4

**A.** For convenience we restrict to Lie-algebras with property  $\mathfrak{g}^\dagger = \mathfrak{g}$ . We will consider general Lagrangians which are (real) quadratic in  $\mathcal{F}_{\mu\nu}$ . Here, in our summation expressions, we write  $\mu < \nu$  instead of  $1 \leq \mu < \nu \leq N$ . Start from the proto Lagrangian

$$\mathcal{G} = \sum_{\mu < \nu, \theta < \rho} h_{(\mu\nu)(\theta\rho)} \text{Tr}[P_{\mu\nu} Q_{\theta\rho\star}] \quad \text{with} \quad \overline{h_{(\mu\nu)(\theta\rho)}} = h_{(\theta\rho)(\mu\nu)} \in \mathbb{C}. \quad (3.16)$$

Note

$$\sum_{\mu < \nu, \theta < \rho} h_{(\mu\nu)(\theta\rho)} \text{Tr}[P_{\mu\nu} P_{\theta\rho}^\dagger] \in \mathbb{R}.$$

For the derivatives of  $\mathcal{G}$  we find,

$$\mathcal{G}^{(\mu\nu)}(\dots, P_{\mu\nu}, \dots; \dots, Q_{\theta\rho\star}, \dots) = \sum_{\alpha < \beta} h_{(\mu\nu)(\alpha\beta)} Q_{\alpha\beta\star}$$

$$\mathcal{G}^{(\theta\rho\star)}(\dots, P_{\mu\nu}, \dots; \dots, Q_{\theta\rho\star}, \dots) = \sum_{\alpha < \beta} h_{(\alpha\beta)(\theta\rho)} P_{\alpha\beta}$$

If we take  $Q_{\theta\rho\star} = P_{\theta\rho}^\dagger$ , one easily checks (3.8),

$$\mathcal{G}^{(\mu\nu)\dagger}(\dots, P_{\mu\nu}, \dots; \dots, P_{\theta\rho}^\dagger, \dots) = \sum_{\alpha < \beta} \overline{h_{(\mu\nu)(\alpha\beta)}} P_{\alpha\beta} = \sum_{\alpha < \beta} h_{(\alpha\beta)(\mu\nu)} P_{\alpha\beta} = \mathcal{G}^{(\mu\nu\star)}.$$

The Lagrangian density

$$\mathcal{G}_A = \sum_{\mu < \nu, \theta < \rho} h_{(\mu\nu)(\theta\rho)} \text{Tr}[\mathcal{F}_{\mu\nu} \mathcal{F}_{\theta\rho}^\dagger], \quad (3.17)$$

can now be put in (3.13) to find the Euler-Lagrange equations. Note however, that  $\mathcal{P}_{\mathfrak{g}}$  cannot be put 'through' the  $h_{(\mu\nu)(\theta\rho)}$  if those are non-real numbers!

So, let us restrict to  $\mathfrak{g}^\dagger = \mathfrak{g}$  and  $h_{(\mu\nu)(\theta\rho)} \in \mathbb{R}$ . Anti-symmetrize  $h_{(\mu\nu)(\theta\rho)}$  to full labels:

$$\hat{h}_{(\mu\nu)(\theta\rho)} = \begin{cases} h_{(\mu\nu)(\theta\rho)} & \text{if } \mu < \nu, \theta < \rho \text{ or } \mu > \nu, \theta > \rho \\ 0 & \text{if } \mu = \nu \text{ and/or } \theta = \rho \\ -h_{(\nu\mu)(\theta\rho)} & \text{if } \mu > \nu, \theta < \rho \\ -h_{(\mu\nu)(\rho\theta)} & \text{if } \mu < \nu, \theta > \rho \end{cases}$$

In this special case

$$\mathcal{G}_A^{(\mu\nu)} = \frac{1}{2} \sum_{\alpha, \beta=1}^N \hat{h}_{(\mu\nu)(\alpha\beta)} \mathcal{F}_{\alpha\beta}^\dagger,$$

and, since  $\mathcal{F}_{\alpha\beta}^\dagger \in \mathfrak{g}$ , the E-L-equations (3.13) become

$$\frac{1}{2} \sum_{\alpha, \beta=1}^N \sum_{\mu=1}^N \hat{h}_{(\mu\kappa)(\alpha\beta)} \left( \partial_\mu \mathcal{F}_{\alpha\beta}^\dagger - \{ \mathcal{A}_\mu, \mathcal{F}_{\alpha\beta}^\dagger \} \right) = 0, \quad 1 \leq \kappa \leq N. \quad (3.18)$$

**B.** For gauge fields on Minkowski space, with coordinates  $x^0, x^1, x^2, x^3$  and metric  $[g^{\mu\nu}] = \text{diag}(1, -1, -1, -1)$ , one usually takes, cf. [DM],

$$h_{(\mu\nu)(\alpha\beta)} = g^{\mu\alpha} g^{\nu\beta} = (-1)^{1+\delta_{\mu 0}} \delta_{\mu\alpha} (-1)^{1+\delta_{\nu 0}} \delta_{\nu\beta} = (-1)^{\delta_{\mu 0} + \delta_{\nu 0}} \delta_{\mu\alpha} \delta_{\nu\beta}.$$

Hence

$$\hat{h}_{(\mu\kappa)(\alpha\beta)} = \text{sgn}(\kappa - \mu) \text{sgn}(\beta - \alpha) (-1)^{\delta_{\mu 0} + \delta_{\kappa 0}} \delta_{\mu\alpha} \delta_{\kappa\beta}.$$

In this special case the Lagrangian density (3.17) reads

$$\mathcal{G}_A = \sum_{0 \leq \mu < \nu \leq 3} (-1)^{\delta_{\mu 0} + \delta_{\nu 0}} \text{Tr} [\mathcal{F}_{\mu\nu} \mathcal{F}_{\mu\nu}^\dagger]. \quad (3.19)$$

The corresponding Euler-Lagrange equations are

$$\sum_{\mu=0}^3 (-1)^{\delta_{\mu 0} + \delta_{\kappa 0}} \nabla_\mu^A \mathcal{F}_{\mu\kappa}^\dagger = 0, \quad 0 \leq \kappa \leq 3. \quad (3.20)$$

For  $\dim \mathfrak{g} = 1$  the term  $\text{ad}_{\mathcal{A}_\mu} \mathcal{F}_{\mu\kappa}^\dagger$  vanishes. This simplification, viz.  $\nabla_\mu^A = \partial_\mu$ , leads to standard electromagnetism in Minkowski space. Indeed, if we put  $\mathcal{A}_0^\dagger = -\Phi$  and  $\text{col}[\mathcal{A}_1^\dagger, \mathcal{A}_2^\dagger, \mathcal{A}_3^\dagger] = \underline{A}$ , then (3.20) turns into Maxwell's equations 'in potential form'

$$\begin{cases} \frac{\partial}{\partial t} \text{div} \underline{A} + \Delta \Phi = 0 \\ \frac{\partial^2}{\partial t^2} \underline{A} - \Delta \underline{A} + \text{grad} \left( \frac{\partial}{\partial t} \Phi + \text{div} \underline{A} \right) = \underline{0} \end{cases} \quad (3.21)$$

If the pair  $\underline{A}, \underline{B}$  satisfies (3.21), then the pair  $\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \text{grad}\Phi$ ,  $\underline{B} = \text{rot}\underline{A}$ , satisfies the classical Maxwell equations.

Finally, imposing the 'Lorenz-Gauge'  $\frac{\partial}{\partial t}\Phi + \text{div}\underline{A} = 0$ , we find the usual wave equations  $\partial_t^2\Phi - \Delta\Phi = 0$ ,  $\partial_t^2\underline{A} - \Delta\underline{A} = \underline{0}$ . For more details see Appendix B.

## 4 Noether Fluxes

'Infinitesimal symmetries' of the Lagrangian density  $\mathcal{L}$  lead to local conservation laws for the solutions of the Euler Lagrange equations. So we are told by Emmy Noether's famous theorem. First we have a short look at the needed concepts as formulated within our special (simple) context.

**Definition 4.1** A *Conservation Law* or *Noether Flux* is a vectorfield on  $\mathbb{R}^N$ , with components  $\mathcal{V}_\psi^\mu$ ,  $1 \leq \mu \leq N$ , which arise from a set of functions of Proto-Lagrangian type,  $\mathcal{V}^\mu$ ,  $1 \leq \mu \leq N$ , cf. (2.13), such that for all solutions  $\Psi$  of the Euler Lagrangian system, cf. Th 2.4, we have

$$\sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \mathcal{V}_\psi^\mu(\underline{x}) = 0, \quad \text{where } \mathcal{V}_\psi^\mu(\underline{x}) = \mathcal{V}^\mu(\Psi(\underline{x}), \Psi^\dagger(\underline{x}), \Psi_{,\mu}(\underline{x}), \Psi_{,\mu}^\dagger(\underline{x}), \underline{x}). \quad (4.1)$$

A conservation law can be named 'trivial' for several reasons: It may happen that for *all* solutions  $\Psi$  the fluxes  $\mathcal{V}_\psi^\mu = 0$ . Another reason for triviality occurs if for all functions  $\Psi$ , whether they are solutions or not, the identity (4.1) is satisfied. For example if the components  $\mathcal{V}_\psi^\mu$  arise from the curl of an arbitrary vector field depending on  $\Psi$ .

Two types of symmetries will be considered here: 'Internal symmetries' and 'External symmetries'. They can be formulated in terms of the *proto-Lagrangian* only.

External symmetries regard transformations of the spatial variables  $\underline{x}$ . We restrict to *affine transforms*.

### Definition 4.2 (Internal symmetries)

A set of linear mappings  $K, L_\mu^\lambda : \mathbb{C}^{r \times c} \rightarrow \mathbb{C}^{r \times c}$ ,  $1 \leq \lambda, \mu \leq N$ , is said to generate an internal (local) symmetry of the proto-Lagrangian  $\mathcal{L}$  if for all  $P, Q_\mu \in \mathbb{C}^{r \times c}$ , all  $\underline{x} \in \mathbb{R}^N$ , and  $s \in \mathbb{R}$ ,  $|s|$  small, one has

$$\begin{aligned} \mathcal{L}(e^{sK}P; (e^{sK}P)^\dagger; \dots e^{sL_\mu^\lambda}Q_\lambda \dots; \dots (e^{sL_\mu^\lambda}Q_\lambda)^\dagger \dots; \underline{x}) = \\ = \mathcal{L}(P; P^\dagger; \dots Q_\mu \dots; \dots Q_\mu^\dagger \dots; \underline{x}) + \mathcal{O}(s^2), \end{aligned} \quad (4.2)$$

In many cases the  $K, L_\mu^\lambda$  are realized by left and/or right multiplication with some fixed matrices in  $\mathbb{C}^{r \times r}$  or  $\mathbb{C}^{c \times c}$ .

Many times there is a special type of internal symmetry which is related to a linear mapping  $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$  in the 'outside world',

$$\begin{aligned} \mathcal{L}(P; P^\dagger; \dots (e^{sA})_\mu^\lambda Q_\lambda \dots; \dots ((e^{sA})_\mu^\lambda Q_\lambda)^\dagger \dots; \underline{x}) = \\ = \mathcal{L}(P; P^\dagger; \dots Q_\mu \dots; \dots Q_\mu^\dagger \dots; \underline{x}) + \mathcal{O}(s^2), \end{aligned} \quad (4.3)$$

**Definition 4.3 (External symmetries)**

The affine mapping  $\underline{x} \mapsto -s\underline{a} + e^{sA}\underline{x}$  on  $\mathbb{R}^N$ , where  $\underline{a} \in \mathbb{R}^N$  and  $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ , a linear mapping, is said to generate an external (local) symmetry of the proto-Lagrangian  $\mathcal{L}$  if for all  $P, Q_\mu \in \mathbb{C}^{r \times c}$ , all  $\underline{x} \in \mathbb{R}^N$ , and  $s \in \mathbb{R}$ ,  $|s|$  small, one has

$$\begin{aligned} \mathcal{L}(P; P^\dagger; \dots Q_\mu \dots; \dots (Q_\mu)^\dagger \dots; -s\underline{a} + e^{sA}\underline{x}) = \\ = \mathcal{L}(P; P^\dagger; \dots Q_\mu \dots; \dots Q_\mu^\dagger \dots; \underline{x}) + \mathcal{O}(s^2). \end{aligned} \quad (4.4)$$

**Remarks 4.4**

- The order constant in  $\mathcal{O}(s^2)$  may depend on all independent variables of  $\mathcal{L}$ .
- If in (4.2)-(4.4) exponents like  $e^{sK}$  are replaced by  $I + sK$  we get equivalent conditions. However in many practical applications the terms  $\mathcal{O}(s^2)$  are identically zero if exponentials are used.
- Local symmetry (4.4) implies

$$\mathcal{L}^{(\nabla)}(P; P^\dagger; \dots Q_\mu \dots; \dots Q_\mu^\dagger \dots; \underline{x}) \cdot (A\underline{x} - \underline{a}) = 0.$$

We now first consider two types of conservation laws in connection with affine transformations in space.

For any vector  $\underline{a} \in \mathbb{R}^N$  we define the *Translation operator*  $\mathbf{T}_{\underline{a}}$  by

$$\mathbf{T}_{\underline{a}}\Psi(\underline{x}) = \Psi(\underline{x} - \underline{a}).$$

For any matrix  $A \in \mathbb{R}^{N \times N}$  we define the *dilation operator*  $\mathbf{R}_A$  by

$$\mathbf{R}_A\Psi(\underline{x}) = \Psi(e^A\underline{x}).$$

**Theorem 4.5**

Suppose that, for some  $K : \mathbb{C}^{r \times c} \rightarrow \mathbb{C}^{r \times c}$  and some  $\underline{a} \in \mathbb{R}^N$ , the proto-Lagrangian  $\mathcal{L}$  has internal local symmetry (4.2) with  $L_\mu^\lambda = \delta_\mu^\lambda K$  and external local symmetry (4.4) with  $A = O$ . Then for any solution  $\Psi$  of the Euler-Lagrange system one has the conservation law

$$\sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \left\{ \text{Tr} \left[ [\mathcal{L}_\psi^{(\mu)}] \cdot (K\Psi - a^\lambda \partial_\lambda \Psi) + [\mathcal{L}_\psi^{(\mu\star)}] \cdot (K\Psi - a^\lambda \partial_\lambda \Psi)^\dagger \right] + a^\mu \mathcal{L}_\psi \right\} = 0. \quad (4.5)$$

**Proof:** By  $\cong$  we mean equality up to a term  $\mathcal{O}(s^2)$ . We study

$$\mathcal{L}(e^{sK}\mathbf{T}_{s\mathbf{a}}\Psi, \mathbf{T}_{s\mathbf{a}}\Psi^\dagger e^{sK^\dagger}, \partial_\mu[e^{sK}\mathbf{T}_{s\mathbf{a}}\Psi], \partial_\mu[\mathbf{T}_{s\mathbf{a}}\Psi^\dagger e^{sK^\dagger}], \underline{x} - s\mathbf{a}).$$

With our conditions it can be written

$$\begin{aligned} \mathcal{L}(e^{sK}\Psi(\underline{x}-s\mathbf{a}); (e^{sK}\Psi(\underline{x}-s\mathbf{a}))^\dagger; \dots \partial_\mu e^{sK}\Psi(\underline{x}-s\mathbf{a}) \dots; \dots \partial_\mu (e^{sK}\Psi(\underline{x}-s\mathbf{a}))^\dagger \dots; \underline{x}-s\mathbf{a}) &\cong \\ \cong \mathcal{L}(\Psi(\underline{x}-s\mathbf{a}); \Psi(\underline{x}-s\mathbf{a})^\dagger; \dots \Psi_{,\mu}(\underline{x}-s\mathbf{a}) \dots; \dots \Psi_{,\mu}(\underline{x}-s\mathbf{a})^\dagger \dots; \underline{x}-s\mathbf{a}) &= \\ = \mathcal{L}_\psi(\underline{x}-s\mathbf{a}) = (\mathbf{T}_{s\mathbf{a}}\mathcal{L}_\psi)(\underline{x}). \end{aligned} \quad (4.6)$$

Differentiate the first line of this at  $s = 0$  and use  $\mathcal{L}^{(\nabla)} \cdot \mathbf{a} = 0$ ,

$$\begin{aligned} \text{Tr}\{[\mathcal{L}_\psi^{(o)}](K\Psi - a^\lambda \partial_\lambda \Psi) + [\mathcal{L}_\psi^{(o*)}](\Psi^\dagger K^\dagger - a^\lambda \partial_\lambda \Psi^\dagger) + \\ + [\mathcal{L}_\psi^{(\mu)}](K\partial_\mu \Psi - a^\lambda \partial_\lambda \partial_\mu \Psi) + [\mathcal{L}_\psi^{(\mu*)}](\partial_\mu \Psi^\dagger K^\dagger - a^\lambda \partial_\lambda \partial_\mu \Psi^\dagger)\}. \end{aligned} \quad (4.7)$$

If  $\Psi$  is a solution we use (2.16) and replace  $[\mathcal{L}_\psi^{(o)}]$  by  $\frac{\partial}{\partial x^\mu}[\mathcal{L}_\psi^{(\mu)}]$ , etc. Now (4.7) can be written as a divergence, which constitutes the left hand side of (4.5), apart from the last term inside  $\{ \}$ . Together with the derivative  $a^\lambda \partial_\lambda \mathcal{L}_\psi = \partial_\mu(a^\mu \mathcal{L}_\psi)$  at  $s = 0$  of the final line of (4.6) we arrive at the wanted conserved current (4.5).  $\blacksquare$

**Example 4.6** Let  $\Gamma^\mu$  and  $M$  be constant complex matrices with  $\Gamma^{\mu\dagger} = \Gamma^\mu$  and  $M = -M^\dagger$ . Then the Lagrangian density

$$\mathcal{L}_\psi = \text{Tr}\{i\Psi^\dagger \Gamma^\mu \partial_\mu \Psi + \Psi^\dagger M \Psi\}, \quad (4.8)$$

for  $\Psi : \mathbb{R}^N \rightarrow \mathbb{C}^{r \times c}$  satisfies the condition of Theorem 4.1 for  $K = O$  and all  $\mathbf{a} \in \mathbb{R}^N$ . The conservation law reads

$$\frac{\partial}{\partial x^\mu} \text{Tr}\{-a^\lambda \Psi^\dagger \Gamma^\mu \partial_\lambda \Psi + a^\mu \Psi^\dagger \Gamma^\lambda \partial_\lambda \Psi + a^\mu \Psi^\dagger M \Psi\} = \frac{\partial}{\partial x^\mu} \text{Tr}\{-a^\lambda \Psi^\dagger \Gamma^\mu \partial_\lambda \Psi\} = 0. \quad (4.9)$$

This can be checked directly for solutions of the PDE:  $\Gamma^\mu \partial_\mu \Psi + M \Psi = 0$ . Observe that in this special case  $\mathcal{L}_\psi = 0$  for solutions.

Also the Lagrangian of Example (2.5b), with *constant* matrices  $K, M, \Gamma^\mu, \mathcal{A}_\mu$  leads to conservation laws of this type.

#### Theorem 4.7

Suppose that, for some  $K : \mathbb{C}^{r \times c} \rightarrow \mathbb{C}^{r \times c}$  and some  $A \in \mathbb{R}^{N \times N}$  with  $\text{Tr} A = 0$ , the proto-Lagrangian  $\mathcal{L}$  has internal local symmetry (4.2) with  $L_\mu^\lambda = K + [A]_\mu^\lambda \mathbf{I}$  and external local symmetry (4.4) with  $\mathbf{a} = \mathbf{0}$ . Then for any solution  $\Psi$  of the Euler-Lagrange system one has the conservation law

$$\begin{aligned} \sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \left\{ \text{Tr} \left[ [\mathcal{L}_\psi^{(\mu)}](K\Psi(\underline{x}) + A_\beta^\alpha x^\beta \Psi_{,\alpha}(\underline{x})) + \right. \right. \\ \left. \left. + [\mathcal{L}_\psi^{(\mu*)}](K\Psi(\underline{x}) + A_\beta^\alpha x^\beta \Psi_{,\alpha}(\underline{x}))^\dagger \right] - A_\beta^\mu x^\beta \mathcal{L}_\psi \right\} = 0. \end{aligned} \quad (4.10)$$

**Proof:** We study

$$\mathcal{L}(e^{sK}\mathbf{R}_{sA}\Psi; \mathbf{R}_{sA}\Psi^\dagger e^{sK^\dagger}; \dots \partial_\mu[e^{sK}\mathbf{R}_{sA}\Psi] \dots; \dots \partial_\mu[\mathbf{R}_{sA}\Psi^\dagger e^{sK^\dagger}] \dots; e^{sA}\underline{x}).$$

With our conditions it can be written,

$$\begin{aligned} & \mathcal{L}(\Psi(e^{sA}\underline{x}); \Psi(e^{sA}\underline{x})^\dagger; \dots \partial_\mu \Psi(e^{sA}\underline{x}) \dots; \dots \partial_\mu \Psi(e^{sA}\underline{x})^\dagger \dots; e^{sA}\underline{x}) \cong \\ & \cong \mathcal{L}(\Psi(e^{sA}\underline{x}); \Psi(e^{sA}\underline{x})^\dagger; \dots (e^{sA})^\lambda_{\mu} \Psi_{,\lambda}(e^{sA}\underline{x}) \dots; \dots (e^{sA})^\lambda_{\mu} \Psi_{,\lambda}(e^{sA}\underline{x})^\dagger \dots; e^{sA}\underline{x}) \cong \\ & \cong \mathcal{L}(\Psi(e^{sA}\underline{x}); \Psi(e^{sA}\underline{x})^\dagger; \dots \Psi_{,\mu}(e^{sA}\underline{x}) \dots; \dots \Psi_{,\mu}(e^{sA}\underline{x})^\dagger \dots; e^{sA}\underline{x}) \cong \\ & \cong \mathcal{L}(\Psi(e^{sA}\underline{x}); \Psi(e^{sA}\underline{x})^\dagger; \dots \Psi_{,\mu}(e^{sA}\underline{x}) \dots; \dots \Psi_{,\mu}(e^{sA}\underline{x})^\dagger \dots; e^{sA}\underline{x}) = \\ & = \mathcal{L}_\psi(e^{sA}\underline{x}) = (\mathbf{R}_{sA}\mathcal{L}_\psi)(\underline{x}). \end{aligned} \quad (4.11)$$

Differentiate the first line of this at  $s = 0$  and use  $\mathcal{L}^{(\nabla)} \cdot A\underline{x} = 0$ :

$$\begin{aligned} & \text{Tr}\{[\mathcal{L}_\psi^{(o)}](K\Psi(\underline{x}) + A_\beta^\alpha x^\beta \Psi_{,\alpha}(\underline{x})) + [\mathcal{L}_\psi^{(\mu)}]\partial_\mu(K\Psi(\underline{x}) + A_\beta^\alpha x^\beta \Psi_{,\alpha}(\underline{x})) + \\ & + [\mathcal{L}_\psi^{(o\star)}](K\Psi(\underline{x}) + A_\beta^\alpha x^\beta \Psi_{,\alpha}(\underline{x}))^\dagger + [\mathcal{L}_\psi^{(\mu\star)}]\partial_\mu(K\Psi(\underline{x}) + A_\beta^\alpha x^\beta \Psi_{,\alpha}(\underline{x}))^\dagger\}. \end{aligned} \quad (4.12)$$

If  $\Psi$  is a solution we use (2.16) and replace  $[\mathcal{L}_\psi^{(o)}]$  by  $\frac{\partial}{\partial x^\mu}[\mathcal{L}_\psi^{(\mu)}]$ , etc. Now (4.12) can be written as a divergence, which constitutes the left hand side of (4.10), apart from the last term between  $\{ \}$ . Together with the derivative at  $s = 0$  of the final line in (4.11):  $A_\beta^\mu \partial_\mu \mathcal{L}_\psi = \partial_\mu(A_\beta^\mu x^\beta \mathcal{L}_\psi)$ , use  $\text{Tr}A = 0$ , we arrive at the conserved current (4.10).  $\blacksquare$

Next we deal with **internal symmetries** only. They play a crucial role in Gauge theories. A simple case first.

#### Theorem 4.8

Suppose that, for some linear  $K : \mathbb{C}^{r \times c} \rightarrow \mathbb{C}^{r \times c}$  the proto-Lagrangian  $\mathcal{L}$  satisfies (4.2) with  $L_\mu^\lambda = \delta_\mu^\lambda K$ . Then for any solution  $\Psi$  of the Euler-Lagrange system one has the conservation law

$$\sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \text{Tr}\{[\mathcal{L}_\psi^{(\mu)}]K\Psi + [\mathcal{L}_\psi^{(\mu\star)}](K\Psi)^\dagger\} = 0, \quad (4.13)$$

**Proof:** Calculate the derivative

$$\frac{\partial}{\partial s} \mathcal{L}(e^{sK}\Psi, (e^{sK}\Psi)^\dagger, \partial_\mu[e^{sK}\Psi], \partial_\mu[e^{sK}\Psi]^\dagger, \underline{x}), \quad \text{at } s = 0.$$

With the notation of (2.5) one finds

$$\text{Tr}\{[\mathcal{L}_\psi^{(o)}][K\Psi] + [\mathcal{L}_\psi^{(o\star)}][K\Psi]^\dagger + [\mathcal{L}_\psi^{(\mu)}][K\Psi_{,\mu}] + [\mathcal{L}_\psi^{(\mu\star)}][K\Psi_{,\mu}]^\dagger\} = 0.$$

If  $\Psi$  happens to be a solution of the Lagrangian system, then with (2.16) this becomes

$$\text{Tr} \left\{ \left[ \frac{\partial}{\partial x^\mu} \mathcal{L}_\psi^{(\mu)} \right] [K\Psi] + \left[ \frac{\partial}{\partial x^\mu} \mathcal{L}_\psi^{(\mu\star)} \right] [K\Psi]^\dagger + [\mathcal{L}_\psi^{(\mu)}] [K\Psi]_{,\mu} + [\mathcal{L}_\psi^{(\mu\star)}] [K\Psi]_{,\mu}^\dagger \right\} = 0,$$

which leads to the wanted 'conserved current', since  $K$  is supposedly constant.  $\blacksquare$

In gauge applications  $K$  is often realized by a right multiplication by some  $A \in \mathbb{C}^{c \times c}$ . In such cases  $K\Psi$  in (4.13) should be replaced by  $\Psi A$ .

All previous considerations can be applied to matrix gauge fields as well if we replace  $\Psi$  by  $\underline{A} = \text{col}[\dots, \mathcal{A}_\mu, \dots]$ . Some subtleties occur however because the range of the functions  $\mathcal{A}_\mu$  is not the whole of  $\mathbb{C}^{c \times c}$  but some real linear subspace  $\mathfrak{g}$  of it. See Appendix A for more details.

This section is concluded with conservation laws for non-commutative free gauge fields which come from the special Lagrangian density (3.8).

#### Theorem 4.9

Consider the proto-Lagrangian  $\mathcal{G}$  of (3.6) with property (3.7) and Lagrange density as denoted in (3.8). For convenience restrict to  $\mathfrak{g} = \mathfrak{g}^\dagger$  only.

a. Suppose  $\mathcal{G}_A^{(\nabla)} \cdot \underline{a} = 0$ , for some  $\underline{a} \in \mathbb{R}^N$  then we have the conservation law

$$\sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \left( \sum_{\kappa=1}^N \text{Re Tr} \left[ \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} : (\underline{a} \cdot \nabla) \mathcal{A}_\kappa \right] - a^\mu \mathcal{G}_A \right) = 0. \quad (4.14)$$

b. If for some  $S = [S_\mu^\lambda] \in \mathbb{R}^{N \times N}$ , with  $\text{Tr} S = 0$ , the assumptions

$$\mathcal{G}_A^{(\nabla)} \cdot S \underline{x} = 0 \quad \text{and} \quad \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \sum_{\alpha=1}^N S_\mu^\alpha \partial_\alpha \mathcal{A}_\nu \right] = 0, \quad (4.15)$$

hold, then we have the conservation law

$$\sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \left( \sum_{\kappa=1}^N 2 \text{Re Tr} \left[ \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} (S \underline{x} \cdot \nabla) \mathcal{A}_\kappa \right] - (S \underline{x} \cdot \underline{e}_\mu) \mathcal{G}_A \right) = 0. \quad (4.16)$$

#### Proof

a. Start from

$$\frac{d}{ds} \mathcal{G}(\dots, \mathcal{F}_{\mu\nu}(\underline{x} - s \underline{a}), \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger(\underline{x} - s \underline{a}), \dots; \underline{x} - s \underline{a}) \Big|_{s=0} = \frac{d}{ds} \mathcal{G}_A(\underline{x} - s \underline{a}) \Big|_{s=0}.$$

Calculate the left hand side with the chain rule and use the assumptions

$$- \sum_{\mu < \nu} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu)} : (\underline{a} \cdot \nabla) \mathcal{F}_{\mu\nu} \right] - \sum_{\mu < \nu} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu\star)} : (\underline{a} \cdot \nabla) \mathcal{F}_{\mu\nu}^\dagger \right] - \underline{a} \cdot \mathcal{G}_A^\nabla =$$

$$= -2\text{Re} \sum_{\mu < \nu} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu)} : (\underline{a} \cdot \nabla) \mathcal{F}_{\mu\nu} \right]. \quad (4.17)$$

With

$$(\underline{a} \cdot \nabla) \mathcal{F}_{\mu\nu} = \partial_\mu (\underline{a} \cdot \nabla \mathcal{A}_\nu) - \partial_\nu (\underline{a} \cdot \nabla \mathcal{A}_\mu) - \{\mathcal{A}_\mu, \underline{a} \cdot \nabla \mathcal{A}_\nu\} + \{\mathcal{A}_\nu, \underline{a} \cdot \nabla \mathcal{A}_\mu\},$$

and the antisymmetries  $\mu \leftrightarrow \nu$ , the expression (4.17) becomes, (mind the hat ^),

$$\begin{aligned} & -\text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \partial_\mu (\underline{a} \cdot \nabla \mathcal{A}_\nu) - \{\mathcal{A}_\mu, \underline{a} \cdot \nabla \mathcal{A}_\nu\} \right] = \\ & -\text{Re} \sum_{\mu, \nu=1}^N \frac{\partial}{\partial x^\mu} \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : (\underline{a} \cdot \nabla \mathcal{A}_\nu) \right] + \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \partial_\mu \hat{\mathcal{G}}_A^{(\mu\nu)} : (\underline{a} \cdot \nabla \mathcal{A}_\nu) + \hat{\mathcal{G}}_A^{(\mu\nu)} : \{\mathcal{A}_\mu, \underline{a} \cdot \nabla \mathcal{A}_\nu\} \right]. \end{aligned}$$

The 2nd term is equal to

$$\text{Re} \sum_{\nu=1}^N \sum_{\mu=1}^N \text{Tr} \left[ \nabla_\mu^A \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\nu)} : (\underline{a} \cdot \nabla \mathcal{A}_\nu) \right] = 0,$$

because of the E-L-equations (3.13).

The right hand side of the 1st formula of this proof equals  $-\partial_\mu (a^\mu \mathcal{L}_A)$ . Hence (4.14).

**b.** Start from

$$\left. \frac{d}{ds} \mathcal{G}(\dots, \mathcal{F}_{\mu\nu}(e^{sS} \underline{x}), \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger(e^{sS} \underline{x}), \dots; e^{sS} \underline{x}) \right|_{s=0} = \left. \frac{d}{ds} \mathcal{G}_A(e^{sS} \underline{x}) \right|_{s=0}.$$

Calculate the left hand side with the chain rule and use  $\mathcal{G}_A^{(\nabla)} \cdot S \underline{x} = 0$ ,

$$\begin{aligned} & 2\text{Re} \sum_{\mu < \nu} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu)} : (S \underline{x} \cdot \nabla) \mathcal{F}_{\mu\nu} \right] = \\ & = \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \partial_\mu ((S \underline{x} \cdot \nabla) \mathcal{A}_\nu) - \{\mathcal{A}_\mu, (S \underline{x} \cdot \nabla) \mathcal{A}_\nu\} - S_\mu^\alpha \partial_\alpha \mathcal{A}_\nu \right]. \end{aligned}$$

Because of the assumption the very final contribution vanishes. Then we proceed as in part **a**. ■

**Note** The orthogonality condition (4.15) is inspired by combining Thm 4.7 with Appendix A. Indeed, another way to obtain the preceding Theorem is to rewrite Thms 4.5, 4.7 in terms of  $\underline{A}$  with the aid of the table in Appendix A.

#### Theorem 4.10

Consider the proto-Lagrangian  $\mathcal{G}$  of (3.6) with property (3.7) and Lagrange density as denoted in (3.8). For convenience consider  $\mathfrak{g} = \mathfrak{g}^\dagger$  only. Suppose  $\mathcal{G}$  satisfies

$$\mathcal{G}(\dots, e^{s\mathcal{B}} P_{\mu\nu} e^{-s\mathcal{B}}, \dots; \dots, e^{-s\mathcal{B}^\dagger} P_{\theta\rho}^\dagger e^{s\mathcal{B}^\dagger}, \dots; \underline{x}) = \mathcal{G}(\dots, P_{\mu\nu}, \dots; \dots, P_{\theta\rho}^\dagger, \dots; \underline{x}), \quad (4.18)$$



for all  $P_{\mu\nu} \in \mathfrak{g} \subset \mathbb{C}^{c \times c}$ ,  $1 \leq \mu < \nu \leq N$ , some fixed  $B \in \mathfrak{g}$  and (small)  $s \in \mathbb{R}$ .

Then, for any solution  $\underline{x} \mapsto \dots \mathcal{A}_\mu(\underline{x}) \dots$  of the Lagrangian system of Theorem 3.3 one has the conservation law

$$\sum_{\mu=1}^N \frac{\partial}{\partial x^\mu} \text{Re} \left( \sum_{\nu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\nu)}] : \{\mathcal{B}, \mathcal{A}_\nu\} \right] \right) = 0. \quad (4.19)$$

**Proof** In (4.18) replace  $P_{\mu\nu} \rightarrow \mathcal{F}_{\mu\nu}$  and  $Q_{\theta\rho} \rightarrow \mathcal{F}_{\theta\rho}^\dagger$  and put the derivative to  $s$  equal to 0 at  $s = 0$ ,

$$\sum_{1 \leq \mu < \nu \leq N} \text{Tr} \left[ [\mathcal{G}_A^{(\mu\nu)}] : (\mathcal{B}\mathcal{F}_{\mu\nu} - \mathcal{F}_{\mu\nu}\mathcal{B}) \right] + \sum_{1 \leq \theta < \rho \leq N} \text{Tr} \left[ [\mathcal{G}_A^{(\theta\rho\star)}] : (-\mathcal{B}^\dagger \mathcal{F}_{\theta\rho}^\dagger + \mathcal{F}_{\theta\rho}^\dagger \mathcal{B}^\dagger) \right] = 0. \quad (4.20)$$

Due to the anti-symmetry in  $\mu \leftrightarrow \nu$  of

$$\mathcal{B}\mathcal{F}_{\mu\nu} - \mathcal{F}_{\mu\nu}\mathcal{B} = \partial_\mu \{\mathcal{B}, \mathcal{A}_\nu\} - \partial_\nu \{\mathcal{B}, \mathcal{A}_\mu\} - \{\mathcal{B}, \{\mathcal{A}_\mu, \mathcal{A}_\nu\}\},$$

applying convention (3.11), together with  $\mathcal{G}_A^{(\mu\nu\star)} = [\mathcal{G}_A^{(\mu\nu)}]^\dagger$ , the 1st term of (4.20) equals the Re-part of

$$\begin{aligned} & \sum_{\mu=1}^N \sum_{\nu=1}^N \text{Tr} \left[ [\hat{\mathcal{G}}_A^{(\mu\nu)}] : (\mathcal{B}\mathcal{F}_{\mu\nu} - \mathcal{F}_{\mu\nu}\mathcal{B}) \right] = \\ &= \sum_{\mu=1}^N \sum_{\nu=1}^N \frac{\partial}{\partial x^\mu} \text{Tr} \left[ [\mathcal{G}_A^{(\mu\nu)}] \{\mathcal{B}, \mathcal{A}_\nu\} \right] - \sum_{\nu=1}^N \sum_{\mu=1}^N \frac{\partial}{\partial x^\nu} \text{Tr} \left[ [\mathcal{G}_A^{(\mu\nu)}] \{\mathcal{B}, \mathcal{A}_\mu\} \right] + \\ & \quad - \sum_{\nu=1}^N \sum_{\mu=1}^N \text{Tr} \left[ [\partial_\mu \mathcal{G}_A^{(\mu\nu)}] \{\mathcal{B}, \mathcal{A}_\nu\} \right] + \sum_{\mu=1}^N \sum_{\nu=1}^N \text{Tr} \left[ [\partial_\nu \mathcal{G}_A^{(\mu\nu)}] \{\mathcal{B}, \mathcal{A}_\mu\} \right] + \\ & \quad - \sum_{\mu=1}^N \sum_{\nu=1}^N \text{Tr} \left[ [\mathcal{G}_A^{(\mu\nu)}] \{\mathcal{B}, \{\mathcal{A}_\mu, \mathcal{A}_\nu\}\} \right]. \end{aligned} \quad (4.21)$$

On the 2nd line we apply the E-L-equations (3.13) together with  $\partial_\nu \mathcal{G}_A^{(\mu\nu)} = -\partial_\nu \mathcal{G}_A^{(\nu\mu)}$ . This together with the 3rd line leads to

$$\begin{aligned} & - \sum_{\nu=1}^N \sum_{\mu=1}^N \text{Tr} \left[ \{\mathcal{A}_\mu, \mathcal{G}_A^{(\mu\nu)}\} \{\mathcal{B}, \mathcal{A}_\nu\} \right] + \sum_{\mu=1}^N \sum_{\nu=1}^N \text{Tr} \left[ \{\mathcal{A}_\nu, \mathcal{G}_A^{(\mu\nu)}\} \{\mathcal{B}, \mathcal{A}_\mu\} \right] + \\ & \quad - \sum_{\mu=1}^N \sum_{\nu=1}^N \text{Tr} \left[ [\mathcal{G}_A^{(\mu\nu)}] \{\mathcal{B}, \{\mathcal{A}_\mu, \mathcal{A}_\nu\}\} \right]. \end{aligned}$$

These 3 terms add up to 0 because for each pair  $\mu, \nu$  separately we can apply the identity

$$- \text{Tr} \left[ \{\mathbf{M}, \mathbf{G}\} : \{\mathbf{B}, \mathbf{N}\} \right] + \text{Tr} \left[ \{\mathbf{N}, \mathbf{G}\} : \{\mathbf{B}, \mathbf{M}\} \right] = \text{Tr} \left[ \mathbf{G} : \{\mathbf{B}, \{\mathbf{M}, \mathbf{N}\}\} \right], \quad (4.22)$$

for matrices  $G, B, M, N \in \mathbb{C}^{r \times r}$ .

(Of course the two terms on the 3rd line of (4.21) are equal. But then, using that equality, the latter trick no longer works for each index pair  $\mu, \nu$  separately!)

Thus we found out that (4.20) corresponds to (4.19). ■

## 5 Static/Dynamic Gauge Extensions of Lagrangians

A basic ingredient for this section is a (fixed) Lie-group  $\mathfrak{G} \subset \mathbb{C}^{c \times c}$  of invertible  $c \times c$ -matrices. Its Lie-algebra  $\mathfrak{g}$  is a  $\mathbb{R}$ -linear subspace of  $\mathbb{C}^{c \times c}$ . Important examples are (subgroups of)  $\mathfrak{G}_J$ , for some fixed invertible matrix  $J \in \mathbb{C}^{c \times c}$ . The relevant definitions are as in section 3,

$$\mathfrak{G}_J = \{ U \in \mathbb{C}^{c \times c} \mid U^\dagger J U = J \}, \quad \mathfrak{g}_J = \{ A \in \mathbb{C}^{c \times c} \mid A^\dagger J + J A = 0 \}. \quad (5.1)$$

In the discussion to follow suitable subspaces of

$$\text{the group } \mathfrak{G}_{\text{loc}} = \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G}) \quad \text{and the } \mathbb{R}\text{-linear space } \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g})$$

will be used. It will be tacitly assumed that the behaviour at  $\infty$  of the considered subspaces is such that our formulae make sense. The  $\mathcal{C}^\infty$ -smoothness condition can often be relaxed. Neither of those assumptions will bother us.

The group action from the right of  $\mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G})$  on  $\mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}^{r \times c})$  is naturally defined by

$$\mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}^{r \times c}) \times \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}^{r \times c}) : (\Psi U)(\underline{x}) = \Psi(\underline{x}) U(\underline{x}).$$

For each  $1 \leq \mu \leq N$ , a group action from the right of  $\mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G})$  on  $\mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g})$  is defined by

$$\mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g}) \times \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g}) : (\mathcal{A}_\mu \triangleleft U)(\underline{x}) = U^{-1}(\underline{x}) \mathcal{A}_\mu(\underline{x}) U(\underline{x}) - U^{-1}(\underline{x}) (\partial_\mu U)(\underline{x}).$$

In the proof of Thm 1.2 it has been shown that this action ('gauge transform') is indeed a (inhomogeneous) group action. This means

$$[\mathcal{A}_\mu \triangleleft U] \triangleleft V = \mathcal{A}_\mu \triangleleft (UV) \quad . \quad (5.2)$$

As before, for given  $\mathcal{A}_\mu, \mathcal{A}_\nu \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g})$ ,  $1 \leq \mu, \nu \leq N$ , define

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \{\mathcal{A}_\mu, \mathcal{A}_\nu\} \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g}). \quad (5.3)$$

Then

$$U^{-1} \mathcal{F}_{\mu\nu} U = \partial_\mu (\mathcal{A}_\nu \triangleleft U) - \partial_\nu (\mathcal{A}_\mu \triangleleft U) - \{(\mathcal{A}_\mu \triangleleft U), (\mathcal{A}_\nu \triangleleft U)\}. \quad (5.4)$$

### Theorem 5.1

Fix a matrix Lie-Group  $\mathfrak{G} \subset \mathbb{C}^{c \times c}$ . Suppose a proto-Lagrangian  $\mathcal{L}$ , cf. (2.13), to be  $\mathfrak{G}$ -invariant, i.e. <sup>3</sup>

$$\forall U \in \mathfrak{G} \quad \forall P \in \mathbb{C}^{r \times c} \quad \forall \underline{R} \in \mathbb{C}^{N \times c} \quad \forall \underline{x} \in \mathbb{R}^N :$$

---

<sup>3</sup>Property (5.5) is named *Global Gauge Invariance* by physicists. The conclusion of Theorem 5.1 is named, in physicists' vernacular, the property of *Local Gauge Invariance*. In mathematicians' jargon however, the usage of 'global', as opposed to 'local', usually refers to a more involved (more difficult) notion.

$$\mathcal{L}(\mathbf{P}\mathbf{U}; \mathbf{U}^\dagger \mathbf{P}^\dagger; \mathbf{R}\mathbf{U}; \mathbf{U}^\dagger \mathbf{R}^\dagger; \underline{x}) = \mathcal{L}(\mathbf{P}; \mathbf{P}^\dagger; \mathbf{R}; \mathbf{R}^\dagger; \underline{x}) \quad (5.5)$$

Then, for all  $\underline{x} \in \mathbb{R}^N$ , the **statically gauge extended Lagrangian density**

$$\mathcal{L}_{\psi, A}(\underline{x}) = \mathcal{L}(\Psi; \Psi^\dagger; \dots, \partial_\mu \Psi + \Psi \mathcal{A}_\mu, \dots; \dots, \partial_\mu \Psi^\dagger + \mathcal{A}_\mu^\dagger \Psi^\dagger, \dots; \underline{x}), \quad (5.6)$$

with any  $\Psi \in \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}^{r \times c})$ ,  $\mathcal{A}_\mu \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{g})$ ,  $1 \leq \mu \leq N$ ,

**equals the statically gauge extended Lagrangian density**

$$\begin{aligned} \mathcal{L}_{\psi \mathcal{U}, A \triangleleft \mathcal{U}}(\underline{x}) &= \\ &= \mathcal{L}(\Psi \mathcal{U}; \mathcal{U}^\dagger \Psi^\dagger; \dots, \partial_\mu (\Psi \mathcal{U}) + (\Psi \mathcal{U})(\mathcal{A}_\mu \triangleleft \mathcal{U}), \dots; \dots, \partial_\mu (\Psi \mathcal{U})^\dagger + (\mathcal{A}_\mu \triangleleft \mathcal{U})^\dagger (\Psi \mathcal{U})^\dagger, \dots; \underline{x}), \\ &\quad \text{with any } \mathcal{U} \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{G}). \end{aligned} \quad (5.7)$$

In (5.6), (5.7) we wrote  $\Psi$  instead of  $\Psi(\underline{x})$ , etc.

**Proof** Straightforward calculation. ■

**Example 5.2** Consider the proto-Lagrangian, cf. (2.13),

$$\mathcal{L}(\mathbf{P}; \mathbf{Q}^\top; \mathbf{R}; \mathbf{S}^\top; \underline{x}) = i \operatorname{Tr}[\mathbf{Q}^\top (\sum_\mu \Gamma^\mu \mathbf{R}_\mu + M \mathbf{P})]$$

with fixed  $\Gamma^\mu, M \in \mathbb{C}^{r \times r}$  and  $[\Gamma^\mu]^\dagger = \Gamma^\mu$ ,  $M^\dagger = -M$ . Put  $\mathfrak{G} = \mathfrak{U}(c) \subset \mathbb{C}^{c \times c}$ , that is the unitary group  $\mathfrak{G}_I$ , with  $I$  the identity matrix. Our proto-Lagrangian is  $\mathfrak{U}(c)$ -invariant

$$i \operatorname{Tr}[\mathbf{U}^\dagger \mathbf{P}^\dagger (\Gamma^\mu \mathbf{R}_\mu \mathbf{U} + M \mathbf{P} \mathbf{U})] = i \operatorname{Tr}[\mathbf{P}^\dagger (\Gamma^\mu \mathbf{R}_\mu + M \mathbf{P})], \quad \mathbf{U} \in \mathfrak{U}(c),$$

because  $\mathbf{U}^\dagger = \mathbf{U}^{-1}$  and the properties of  $\operatorname{Tr}$ .

Then the statically extended Lagrangian density

$$\mathcal{L}_{\psi, A}(\underline{x}) = i \operatorname{Tr}[\Psi^\dagger (\Gamma^\mu (\partial_\mu \Psi + \Psi \mathcal{A}_\mu) + M \Psi)], \quad (5.8)$$

with any  $\Psi \in \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}^{r \times c})$ ,  $\mathcal{A}_\mu \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{u}(c))$ ,  $1 \leq \mu \leq N$ ,

**equals the statically extended Lagrangian density**

$$\mathcal{L}_{\psi \mathcal{U}, A \triangleleft \mathcal{U}}(\underline{x}) = i \operatorname{Tr}[\mathcal{U}^\dagger \Psi^\dagger (\Gamma^\mu (\partial_\mu (\Psi \mathcal{U}) + \Psi \mathcal{U} (\mathcal{U}^{-1} \mathcal{A}_\mu \mathcal{U} - \mathcal{U}^{-1} \partial_\mu \mathcal{U})) + M \Psi \mathcal{U})], \quad (5.9)$$

with any  $\mathcal{U} \in \mathcal{C}^\infty(\mathbb{R}^N; \mathfrak{U}(c))$ .

Note that, if  $M$  is replaced by the 'nonlinearity'  $i \Psi \Psi^\dagger$ , the argument still holds. ■

### Theorem 5.3

- Suppose that the statically gauge extended Lagrange density  $\mathcal{L}_{\psi,A}$ , cf. (5.6) leads to an  $\mathbb{R}$ -valued Langrangian functional  $\mathcal{L}_{\psi,A}$ . The E-L-equations are

$$\begin{aligned} \mathcal{L}_{\psi,A}^{(o)} - \sum_{\mu=1}^N \left( \frac{\partial}{\partial x^\mu} [\mathcal{L}_{\psi,A}^{(\mu)}] - [\mathcal{A}_\mu \mathcal{L}_{\psi,A}^{(\mu)}] \right) &= 0, \\ \mathcal{P}_{\mathfrak{g}} \left( \Psi^\dagger [\mathcal{L}_{\psi,A}^{(\kappa)\dagger} + \mathcal{L}_{\psi,A}^{(\kappa\star)}] \right) &= 0, \quad \mathcal{P}_{\mathfrak{g}} \left( \frac{\Psi^\dagger [\mathcal{L}_{\psi,A}^{(\kappa)\dagger} - \mathcal{L}_{\psi,A}^{(\kappa\star)}]}{i} \right) = 0, \quad 1 \leq \kappa \leq N. \end{aligned} \quad (5.10)$$

Here  $\mathcal{P}_{\mathfrak{g}} : \mathbb{C}^{c \times c} \rightarrow \mathbb{C}^{c \times c}$  denotes the  $\mathbb{R}$ -orthogonal projection on  $\mathfrak{g}$ .

- If it happens that  $\mathcal{P}_{\mathfrak{g}}(iZ) = i\mathcal{P}_{\mathfrak{g}}^\perp Z$ ,  $Z \in \mathbb{C}^{c \times c}$ , the 2nd line in (5.10) reduces to

$$\Psi^\dagger \mathcal{L}_{\psi,A}^{(\kappa)\dagger} + (\mathcal{P}_{\mathfrak{g}} - \mathcal{P}_{\mathfrak{g}}^\perp) \Psi^\dagger \mathcal{L}_{\psi,A}^{(\kappa\star)} = 0, \quad 1 \leq \kappa \leq N. \quad (5.11)$$

- In the important special case  $\mathfrak{g} = \mathfrak{g}_J$ , with  $J = J^\dagger = J^{-1}$ , (5.11) can be written

$$\mathcal{L}_{\psi,A}^{(\kappa)} \Psi - J \Psi^\dagger \mathcal{L}_{\psi,A}^{(\kappa\star)} J = 0, \quad 1 \leq \kappa \leq N. \quad (5.12)$$

**Proof** • The perturbed statically extended Lagrangian  $\mathcal{L}_{\psi,A}$  reads

$$\begin{aligned} \mathcal{L}(\Psi + \varepsilon \mathbf{H}; \Psi^\dagger + \varepsilon^* \mathbf{K}; \dots, \partial_\mu(\Psi + \varepsilon \mathbf{H}) + (\Psi + \varepsilon \mathbf{H})(\mathcal{A}_\mu + \varepsilon_\kappa \delta_{\mu\kappa} \mathcal{H}), \dots; \\ ; \dots, \partial_\mu(\Psi^\dagger + \varepsilon^* \mathbf{K}) + (\mathcal{A}_\mu^\dagger + \varepsilon_\kappa \delta_{\mu\kappa} \mathcal{H}^\dagger)(\Psi^\dagger + \varepsilon^* \mathbf{K}), \dots; \underline{x}) \end{aligned}$$

The results of  $\frac{d}{d\varepsilon} \Big|_{\varepsilon=0}$ ,  $\frac{d}{d\varepsilon^*} \Big|_{\varepsilon^*=0}$ ,  $\frac{d}{d\varepsilon_\kappa} \Big|_{\varepsilon_\kappa=0}$ ,  $1 \leq \kappa \leq N$ , being put to 0 are, for all functions  $\mathbf{H}, \mathbf{K}, \mathcal{H}$ ,

$$\begin{aligned} \text{Tr}[\mathcal{L}^{(o)} : \mathbf{H}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu)} : \partial_\mu \mathbf{H}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu)} : \mathbf{H} \mathcal{A}_\mu] &= 0, \\ \text{Tr}[\mathcal{L}^{(o\star)} : \mathbf{K}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu\star)} : \partial_\mu \mathbf{K}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu\star)} : \mathcal{A}_\mu^\dagger \mathbf{K}] &= 0, \\ \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu)} : \Psi \delta_{\mu\kappa} \mathcal{H}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu\star)} : \delta_{\mu\kappa} \mathcal{H}^\dagger \Psi^\dagger] &= 0, \quad 1 \leq \kappa \leq N. \end{aligned}$$

The usual partial integration techniques applied to the first two lines lead to the E-L-equations for  $\Psi$ . Also use Theorem 2.4.

From the final line we arrive at (5.10) because of the trace identity

$$\text{Tr}[XZ + YZ^\dagger] = \text{Re Tr} \left[ \left( X^\dagger + Y \right)^\dagger Z \right] - i \text{Re Tr} \left[ \left( \frac{X^\dagger - Y}{i} \right)^\dagger Z \right]. \quad (5.13)$$

- If for  $X, Y \in \mathbb{C}^{c \times c}$  one has  $\mathcal{P}_{\mathfrak{g}}(X + Y) = 0$  and  $\mathcal{P}_{\mathfrak{g}}^\perp(X - Y) = 0$ , it follows that  $X + (\mathcal{P}_{\mathfrak{g}} - \mathcal{P}_{\mathfrak{g}}^\perp)Y = 0$  and also  $Y + (\mathcal{P}_{\mathfrak{g}} - \mathcal{P}_{\mathfrak{g}}^\perp)X = 0$ .
- In this special case  $(\mathcal{P}_{\mathfrak{g}} - \mathcal{P}_{\mathfrak{g}}^\perp)Y = -JY^\dagger J$  and  $\mathcal{P}_{\mathfrak{g}}[Y^\dagger] = [\mathcal{P}_{\mathfrak{g}}Y]^\dagger$ . ■

#### Examples 5.4

Note that in the E-L-equations (5.10) the  $\mathcal{A}_\mu$  occur only 'algebraically'.

The  $\partial_\mu \mathcal{A}$  are not involved!

**a.** For the Lagrangian densities from examples 2.5a and 5.2 the 2nd set of E-L-equations (5.12) does not depend on  $\mathcal{A}$ . If we choose  $\mathfrak{g} = \mathfrak{g}_J$ , the 2nd line reads

$$\Psi^\dagger \Gamma^\kappa \Psi = 0, \quad 1 \leq \kappa \leq N.$$

It means that  $\Psi$  can only take values in a cone in  $\mathbb{C}^{r \times c}$ . If one of the  $\Gamma^\kappa = \Gamma^{\kappa\dagger}$  is strictly positive, the only solutions are  $\Psi = 0$ , the trivial ones. If a nontrivial choice for  $\Psi$  is possible it can be substituted in the 1st E-L-equation and we are left with an algebraic equation for the  $\mathcal{A}_\kappa$ .

**b.** For the Lagrangian densities from example 2.5c, again with  $\mathfrak{g} = \mathfrak{g}_J$ , the 2nd set of E-L-equations becomes

$$\sum_{\mu=1}^N [\partial_\mu \Psi + \Psi \mathcal{A}_\mu]^\dagger \Theta^{\mu\kappa} \Psi - J \left( \sum_{\mu=1}^N [\Psi^\dagger \Theta^{\kappa\mu} [\partial_\mu \Psi + \Psi \mathcal{A}_\mu]] \right) J = 0, \quad 1 \leq \kappa \leq N,$$

which is algebraic in the  $\mathcal{A}_\kappa$ . ■

Finally we want to consider the **dynamically gauge extended** Lagrangian density or **Gauge field extended** Lagrangian density of type  $\mathcal{L}_{\psi,A}(\underline{x}) + \mathcal{G}_A(\underline{x})$ .

#### Theorem 5.5

Fix a matrix Liegroup  $\mathfrak{G} \subset \mathbb{C}^{c \times c}$  with Lie algebra  $\mathfrak{g} \subset \mathbb{C}^{c \times c}$  and property  $\mathfrak{g}^\dagger = \mathfrak{g}$ .

Fix a proto Lagrangian of type (2.13)

$$(P; Q^\top; \underline{R}; \underline{S}^\top; \underline{x}) \mapsto \mathcal{L}(P; Q^\top; \underline{R}; \underline{S}^\top; \underline{x}),$$

leading to a  $\mathbb{R}$ -valued Lagrangian functional  $\mathcal{L}$ . Require the special property

$$\forall P \forall \underline{R} \forall \underline{x}: \mathcal{P}_{\mathfrak{g}} \left( \frac{P^\dagger [\mathcal{L}^{(\kappa)\dagger}(P; P^\dagger; \underline{R}; \underline{R}^\dagger; \underline{x}) - \mathcal{L}^{(\kappa\star)}(P; P^\dagger; \underline{R}; \underline{R}^\dagger; \underline{x})]}{\mathfrak{i}} \right) = 0. \quad (5.14)$$

Fix a second proto Lagrangian of type (3.6) and such that

$$\forall R_{\mu\nu} \in \mathfrak{g} : \mathcal{G}(\dots, R_{\mu\nu}, \dots; \dots, R_{\theta\rho}^\dagger, \dots; \underline{x}) \in \mathbb{R}.$$

Consider the **dynamically extended** Lagrangian density

$$\begin{aligned} \mathcal{L}_{\psi,A}(\underline{x}) + \mathcal{G}_A(\underline{x}) &= \mathcal{L}(\Psi; \Psi^\dagger; \dots, \partial_\mu \Psi + \Psi \mathcal{A}_\mu, \dots; \dots, \partial_\mu \Psi^\dagger + \mathcal{A}_\mu^\dagger \Psi^\dagger, \dots; \underline{x}) + \\ &+ \mathcal{G}(\dots, \mathcal{F}_{\mu\nu}(\underline{x}), \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger(\underline{x}), \dots; \underline{x}) \end{aligned} \quad (5.15)$$

with any  $\Psi \in \mathcal{C}^\infty(\mathbb{R}^N : \mathbb{C}^{r \times c})$ ,  $\mathcal{A}_\mu \in \mathcal{C}^\infty(\mathbb{R}^N : \mathfrak{g})$ ,  $1 \leq \mu \leq N$ .

- The Euler-Lagrange equations are, with  $\mathcal{L}_{\psi,A}^{(o)}$  instead of  $\mathcal{L}_{\psi,A}^{(o)}(\underline{x})$ , etc.,

$$\begin{aligned} & [\mathcal{L}_{\psi,A}^{(o)}] - \sum_{\mu=1}^N \left( \frac{\partial}{\partial x^\mu} [\mathcal{L}_{\psi,A}^{(\mu)}] - [\mathcal{A}_\mu \mathcal{L}_{\psi,A}^{(\mu)}] \right) = 0, \\ & \mathcal{P}_{\mathfrak{g}} \left( \Psi^\dagger [\mathcal{L}_{\psi,A}^{(\kappa)\dagger} + \mathcal{L}_{\psi,A}^{(\kappa\star)}] \right) - 2 \sum_{\mu=1}^N \left( \partial_\mu \mathcal{P}_{\mathfrak{g}} [\hat{\mathcal{G}}_A^{(\mu\kappa)}] - \{ \mathcal{A}_\mu, \mathcal{P}_{\mathfrak{g}} [\hat{\mathcal{G}}_A^{(\mu\kappa)}] \} \right)^\dagger = 0, \quad 1 \leq \kappa \leq N. \end{aligned} \quad (5.16)$$

Here  $\mathcal{P}_{\mathfrak{g}} : \mathbb{C}^{c \times c} \rightarrow \mathbb{C}^{c \times c}$  denotes the  $\mathbb{R}$ -orthogonal projection on  $\mathfrak{g}$ .

- In the special case  $\mathfrak{g} = \mathfrak{g}_J$ , with  $J = J^\dagger = J^{-1}$ , the 2nd line in (5.16) can be rewritten

$$\mathcal{L}_{\psi,A}^{(\kappa)} \Psi - J \Psi^\dagger \mathcal{L}_{\psi,A}^{(\kappa\star)} J - 2 \sum_{\mu=1}^N \left( \partial_\mu \mathcal{P}_{\mathfrak{g}} [\hat{\mathcal{G}}_A^{(\mu\kappa)}] - \{ \mathcal{A}_\mu, \mathcal{P}_{\mathfrak{g}} [\hat{\mathcal{G}}_A^{(\mu\kappa)}] \} \right) = 0, \quad 1 \leq \kappa \leq N. \quad (5.17)$$

**Proof** • The perturbed gauge supplemented Lagrangian reads

$$\begin{aligned} & \mathcal{L}(\Psi + \varepsilon \mathbf{H}; \Psi^\dagger + \varepsilon^* \mathbf{K}; \dots, \partial_\mu(\Psi + \varepsilon \mathbf{H}) + (\Psi + \varepsilon \mathbf{H})(\mathcal{A}_\mu + \varepsilon_\kappa \delta_{\mu\kappa} \mathcal{H}), \dots; \\ & \quad ; \dots, \partial_\mu(\Psi^\dagger + \varepsilon^* \mathbf{K}) + (\mathcal{A}_\mu^\dagger + \varepsilon_\kappa \delta_{\mu\kappa} \mathcal{H}^\dagger)(\Psi^\dagger + \varepsilon^* \mathbf{K}), \dots; \underline{x}) + \\ & \quad + \mathcal{G}(\dots, \mathcal{F}_{\mu\nu, \varepsilon\kappa}, \dots; \dots, \mathcal{F}_{\theta\rho, \varepsilon\kappa}^\dagger, \dots; \underline{x}), \quad 1 \leq \kappa \leq N, \end{aligned}$$

where

$$\mathcal{F}_{\mu\nu, \varepsilon, \kappa} = \mathcal{F}_{\mu\nu} + \varepsilon_\kappa \delta_{\nu\kappa} \left[ \partial_\mu \mathcal{H} - \{ \mathcal{A}_\mu, \mathcal{H} \} \right] - \varepsilon_\kappa \delta_{\mu\kappa} \left[ \partial_\nu \mathcal{H} - \{ \mathcal{A}_\nu, \mathcal{H} \} \right],$$

The results of  $\frac{d}{d\varepsilon} \Big|_{\varepsilon=0}$ ,  $\frac{d}{d\varepsilon^*} \Big|_{\varepsilon^*=0}$ ,  $\frac{d}{d\varepsilon_\kappa} \Big|_{\varepsilon_\kappa=0}$ , being put to 0 are, respectively,

$$\begin{aligned} & \text{Tr}[\mathcal{L}^{(o)} : \mathbf{H}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu)} : \partial_\mu \mathbf{H}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu)} : \mathbf{H} \mathcal{A}_\mu] = 0, \\ & \text{Tr}[\mathcal{L}^{(o\star)} : \mathbf{K}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu\star)} : \partial_\mu \mathbf{K}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu\star)} : \mathcal{A}_\mu^\dagger \mathbf{K}] = 0, \\ & \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu)} : \Psi \delta_{\mu\kappa} \mathcal{H}] + \sum_{\mu} \text{Tr}[\mathcal{L}^{(\mu\star)} : \delta_{\mu\kappa} \mathcal{H}^\dagger \Psi^\dagger] + \\ & \quad - 2 \sum_{\mu} \text{Re Tr} \left[ \left( \mathcal{P}_{\mathfrak{g}} \partial_\mu \hat{\mathcal{G}}_A^{(\mu\kappa\star)} + \mathcal{P}_{\mathfrak{g}} \{ \mathcal{A}_\mu^\dagger, \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa\star)} \} \right)^\dagger [\mathcal{H}] \right] = 0, \quad 1 \leq \kappa \leq N. \end{aligned}$$

With (5.13) the 3rd set of equations can be rewritten

$$\text{Re Tr} \left[ \left( \Psi^\dagger ([\mathcal{L}^{(\kappa)}]^\dagger + [\mathcal{L}^{(\kappa\star)}])^\dagger \mathcal{H} \right) + \text{i Re Tr} \left[ \left( \text{i} \Psi^\dagger ([\mathcal{L}^{(\kappa)}]^\dagger - [\mathcal{L}^{(\kappa\star)}])^\dagger \mathcal{H} \right) + \right.$$

$$- 2 \sum_{\mu} \text{Re Tr} \left[ \left( \mathcal{P}_{\mathfrak{g}} \partial_{\mu} \hat{\mathcal{G}}_A^{(\mu\kappa)} - \{ \mathcal{A}_{\mu}, \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} \} \right)^{\dagger\dagger} [\mathcal{H}] \right] = 0, \quad 1 \leq \kappa \leq N.$$

Because of assumption (5.14) the  $\text{iRe Tr}$ -term cancels. The assumption  $\mathfrak{g}^{\dagger} = \mathfrak{g}$  enables us to interchange  $\dagger$  and  $\mathcal{P}_{\mathfrak{g}}$ .

- Finally (5.17) follows as in the proof of Thm (5.3). ■

Finally we want to find the conservation law of 'conserved currents'.

### Theorem 5.6

Consider proto-Lagrangians  $\mathcal{L}$  and  $\mathcal{G}$  as in Theorem 5.5. Suppose for some  $\mathcal{B} \in \mathfrak{g}$  they both have the invariance properties

$$\begin{aligned} \mathcal{L}(\text{P}e^{s\mathcal{B}}; (\text{P}e^{s\mathcal{B}})^{\dagger}; \dots \text{Q}_{\lambda}e^{s\mathcal{B}} \dots; \dots (\text{Q}_{\lambda}e^{s\mathcal{B}})^{\dagger} \dots; \underline{x}) = \\ = \mathcal{L}(\text{P}; \text{P}^{\dagger}; \dots \text{Q}_{\lambda} \dots; \dots \text{Q}_{\lambda}^{\dagger} \dots; \underline{x}) + \mathcal{O}(s^2), \end{aligned} \quad (5.18)$$

$$\begin{aligned} \mathcal{G}(\dots, e^{-s\mathcal{B}} \text{R}_{\mu\nu} e^{s\mathcal{B}}, \dots; \dots, e^{s\mathcal{B}^{\dagger}} \text{R}_{\theta\rho}^{\dagger} e^{-s\mathcal{B}^{\dagger}}, \dots; \underline{x}) = \\ = \mathcal{G}(\dots, \text{R}_{\mu\nu}, \dots; \dots, \text{R}_{\theta\rho}^{\dagger}, \dots; \underline{x}) + \mathcal{O}(s^2). \end{aligned} \quad (5.19)$$

Then, the solutions to the E-L-system (5.16) satisfy the conservation law

$$\sum_{\mu=1}^N \frac{\partial}{\partial x_{\mu}} \left\{ \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu)} : \Psi \mathcal{B} \right] + \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu\star)} : \mathcal{B}^{\dagger} \Psi^{\dagger} \right] + \sum_{\kappa=1}^N 2 \text{Re Tr} \left[ \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} : \{ \mathcal{A}_{\kappa}, \mathcal{B} \} \right] \right\} = 0. \quad (5.20)$$

**Proof** Add the Lagrange densities  $\mathcal{L}_{\psi,A}$  and  $\mathcal{G}_A$  and put to 0 the  $\frac{d}{ds}$  of the expression

$$\begin{aligned} \mathcal{L}(\Psi e^{s\mathcal{B}}; e^{s\mathcal{B}^{\dagger}} \Psi^{\dagger}; \dots, \partial_{\mu} \Psi e^{s\mathcal{B}} + \Psi \mathcal{A}_{\mu} e^{s\mathcal{B}}, \dots; \dots, e^{s\mathcal{B}^{\dagger}} \partial_{\mu} \Psi^{\dagger} + e^{s\mathcal{B}^{\dagger}} \mathcal{A}_{\mu}^{\dagger} \Psi^{\dagger}, \dots; \underline{x}) + \\ + \mathcal{G}(\dots, e^{-s\mathcal{B}} \mathcal{F}_{\mu\nu} e^{s\mathcal{B}}, \dots; \dots, e^{s\mathcal{B}^{\dagger}} \mathcal{F}_{\theta\rho}^{\dagger} e^{-s\mathcal{B}^{\dagger}}, \dots; \underline{x}) \end{aligned}$$

One finds,

$$\begin{aligned} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(o)} : \Psi \mathcal{B} \right] + \sum_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu)} : \partial_{\mu} \Psi \mathcal{B} \right] + \sum_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu)} : \Psi \mathcal{A}_{\mu} \mathcal{B} \right] + \\ + \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(o\star)} : \mathcal{B}^{\dagger} \Psi^{\dagger} \right] + \sum_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu\star)} : \mathcal{B}^{\dagger} \partial_{\mu} \Psi^{\dagger} \right] + \sum_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu\star)} : \mathcal{B}^{\dagger} \mathcal{A}_{\mu}^{\dagger} \Psi^{\dagger} \right] + \\ + \sum_{\mu < \nu} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu)} : \{ \mathcal{F}_{\mu\nu}, \mathcal{B} \} \right] + \sum_{\theta < \rho} \text{Tr} \left[ \mathcal{G}_A^{(\theta\rho\star)} : \{ \mathcal{B}^{\dagger}, \mathcal{F}_{\theta\rho}^{\dagger} \} \right] = 0. \end{aligned} \quad (5.21)$$

Rewrite the 3rd term and the 6th term:

$$\begin{aligned}\sum_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu)} : \Psi \mathcal{A}_{\mu} \mathcal{B} \right] &= \sum_{\kappa} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\kappa)} : \Psi \{ \mathcal{A}_{\kappa}, \mathcal{B} \} \right] + \sum_{\mu} \text{Tr} \left[ \mathcal{A}_{\mu} \mathcal{L}_{\psi,A}^{(\mu)} : \Psi \mathcal{B} \right], \\ \sum_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu\star)} : (\Psi \mathcal{A}_{\mu} \mathcal{B})^{\dagger} \right] &= \sum_{\kappa} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\kappa\star)} : (\Psi \{ \mathcal{A}_{\kappa}, \mathcal{B} \})^{\dagger} \right] + \sum_{\mu} \text{Tr} \left[ \mathcal{A}_{\mu}^{\dagger} \mathcal{L}_{\psi,A}^{(\mu\star)} : (\Psi \mathcal{B})^{\dagger} \right].\end{aligned}$$

These identities, together with the 1st E-L-equation of (5.16) turn the first 6 terms of (5.21) into

$$\begin{aligned}\sum_{\mu} \partial_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu)} : \Psi \mathcal{B} \right] &+ \sum_{\mu} \partial_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu\star)} : \mathcal{B}^{\dagger} \Psi^{\dagger} \right] + \\ &+ \sum_{\kappa} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\kappa)} : \Psi \{ \mathcal{A}_{\kappa}, \mathcal{B} \} \right] + \sum_{\kappa} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\kappa\star)} : (\Psi \{ \mathcal{A}_{\kappa}, \mathcal{B} \})^{\dagger} \right]\end{aligned}$$

With Trace identity (5.13) and condition (5.14) the latter becomes

$$\begin{aligned}\sum_{\mu} \partial_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu)} : \Psi \mathcal{B} \right] &+ \sum_{\mu} \partial_{\mu} \text{Tr} \left[ \mathcal{L}_{\psi,A}^{(\mu\star)} : \mathcal{B}^{\dagger} \Psi^{\dagger} \right] + \\ &+ 2 \sum_{\kappa, \mu=1}^N \text{Re} \text{Tr} \left[ \left( \mathcal{P}_{\mathfrak{g}} \partial_{\mu} \hat{\mathcal{G}}_A^{(\mu\kappa)} - \{ \mathcal{A}_{\mu}, \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} \} \right) : \{ \mathcal{A}_{\kappa}, \mathcal{B} \} \right].\end{aligned}\quad (5.22)$$

Next, because of (anti)symmetry,  $\mathcal{B} \in \mathfrak{g}$  being constant and the definition of  $\mathcal{F}_{\mu\nu}$ , the final 2 terms of (5.21) equal to

$$\begin{aligned}\text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \{ \mathcal{F}_{\mu\nu}, \mathcal{B} \} \right] &= \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \partial_{\mu} \{ \mathcal{A}_{\nu}, \mathcal{B} \} \right] + \\ &- \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \partial_{\nu} \{ \mathcal{A}_{\mu}, \mathcal{B} \} \right] - \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \{ \{ \mathcal{A}_{\mu}, \mathcal{A}_{\nu} \}, \mathcal{B} \} \right] = \\ &= 2 \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \partial_{\mu} \{ \mathcal{A}_{\nu}, \mathcal{B} \} \right] - \text{Re} \sum_{\mu, \nu=1}^N \text{Tr} \left[ \hat{\mathcal{G}}_A^{(\mu\nu)} : \{ \{ \mathcal{A}_{\mu}, \mathcal{A}_{\nu} \}, \mathcal{B} \} \right].\end{aligned}\quad (5.23)$$

If we add (5.22), (5.23), we arrive at (5.20), up to a term

$$- \text{Re} \sum_{\kappa, \mu=1}^N \left( 2 \text{Tr} \left[ \{ \mathcal{A}_{\mu}, \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} \} : \{ \mathcal{A}_{\kappa}, \mathcal{B} \} \right] + \text{Tr} \left[ \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} : \{ \{ \mathcal{A}_{\mu}, \mathcal{A}_{\kappa} \}, \mathcal{B} \} \right] \right).$$



Split the first term in this summation. It becomes,

$$\begin{aligned}
& - \operatorname{Re} \sum_{\kappa, \mu=1}^N \left( \operatorname{Tr} \left[ \{ \mathcal{A}_\mu, \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} \} : \{ \mathcal{A}_\kappa, \mathcal{B} \} \right] - \operatorname{Tr} \left[ \{ \mathcal{A}_\kappa, \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} \} : \{ \mathcal{A}_\mu, \mathcal{B} \} \right] + \right. \\
& \quad \left. + \operatorname{Tr} \left[ \mathcal{P}_{\mathfrak{g}} \hat{\mathcal{G}}_A^{(\mu\kappa)} : \{ \{ \mathcal{A}_\mu, \mathcal{A}_\kappa \}, \mathcal{B} \} \right] \right).
\end{aligned}$$

Each term in this sum equals 0 because of the trace identity

$$\operatorname{Tr} \left[ \{ \mathbf{M}, \mathbf{G} \} : \{ \mathbf{K}, \mathbf{B} \} \right] - \operatorname{Tr} \left[ \{ \mathbf{K}, \mathbf{G} \} : \{ \mathbf{M}, \mathbf{B} \} \right] + \operatorname{Tr} \left[ \mathbf{G} : \{ \{ \mathbf{M}, \mathbf{K} \}, \mathbf{B} \} \right] = 0.$$

Indeed, note that for any  $\mathbf{M}, \mathbf{G}, \mathbf{K}, \mathbf{B} \in \mathbb{C}^{c \times c}$ ,

$$\begin{aligned}
& \operatorname{Tr} \left[ \mathbf{M} \mathbf{G} \mathbf{K} \mathbf{B} - \mathbf{G} \mathbf{M} \mathbf{K} \mathbf{B} - \mathbf{M} \mathbf{G} \mathbf{B} \mathbf{K} + \mathbf{G} \mathbf{M} \mathbf{B} \mathbf{K} - \mathbf{K} \mathbf{G} \mathbf{M} \mathbf{B} + \mathbf{G} \mathbf{K} \mathbf{M} \mathbf{B} + \right. \\
& \quad \left. + \mathbf{K} \mathbf{G} \mathbf{B} \mathbf{M} - \mathbf{G} \mathbf{K} \mathbf{B} \mathbf{M} + \mathbf{G} \mathbf{M} \mathbf{K} \mathbf{B} - \mathbf{G} \mathbf{K} \mathbf{M} \mathbf{B} - \mathbf{G} \mathbf{B} \mathbf{M} \mathbf{K} + \mathbf{G} \mathbf{B} \mathbf{K} \mathbf{M} \right] = 0.
\end{aligned}$$

■

## A Addendum on Free Gauge Fields

If we put

$$\begin{aligned}
\mathcal{G}_A(\underline{x}) &= \mathcal{G}(\dots, \mathcal{F}_{\mu\nu}(\underline{x}), \dots; \dots, \mathcal{F}_{\theta\rho}^\dagger(\underline{x}), \dots; \underline{x}) = \\
&= \mathcal{G}(\dots, \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \{\mathcal{A}_\mu, \mathcal{A}_\nu\}, \dots; \dots, \partial_\mu \mathcal{A}_\nu^\dagger - \partial_\nu \mathcal{A}_\mu^\dagger + \{\mathcal{A}_\mu^\dagger, \mathcal{A}_\nu^\dagger\}, \dots; \underline{x}) = \\
&= \mathcal{L}(\underline{\mathcal{A}}(\underline{x}); \underline{\mathcal{A}}^\dagger(\underline{x}); \dots, \partial_\mu \underline{\mathcal{A}}(\underline{x}), \dots; \dots, \partial_\mu \underline{\mathcal{A}}^\dagger(\underline{x}), \dots; \underline{x}), \quad (\text{A.1})
\end{aligned}$$

with  $\underline{\mathcal{A}} = \text{col}[\dots, \mathcal{A}_\mu, \dots]$ , which now plays the role of  $\Psi$  in section 2, we get, in accordance with our notation in section 2,

$$\begin{aligned}
\mathcal{L}_A^{(\circ)} &= \text{row} \left[ \dots \dots \dots - \sum_{\mu=1}^N \{\hat{\mathcal{G}}_A^{(\mu\kappa)}, \mathcal{A}_\mu\} \dots \dots \right] \\
\mathcal{L}_A^{(1)} &= \text{row} \left[ 0 \quad \mathcal{G}_A^{(12)} \quad \mathcal{G}_A^{(13)} \quad \dots \quad \mathcal{G}_A^{(1\kappa)} \quad \dots \quad \mathcal{G}_A^{(1N)} \right] \\
\mathcal{L}_A^{(2)} &= \text{row} \left[ -\mathcal{G}_A^{(12)} \quad 0 \quad \mathcal{G}_A^{(23)} \quad \dots \quad \mathcal{G}_A^{(2\kappa)} \quad \dots \quad \mathcal{G}_A^{(2N)} \right] \\
\mathcal{L}_A^{(3)} &= \text{row} \left[ -\mathcal{G}_A^{(13)} \quad -\mathcal{G}_A^{(23)} \quad 0 \quad \dots \quad \mathcal{G}_A^{(3\kappa)} \quad \dots \quad \mathcal{G}_A^{(3N)} \right] \\
\dots &= \text{row} \left[ \dots \dots \dots \dots \dots \dots \dots \right] \\
\mathcal{L}_A^{(\kappa)} &= \text{row} \left[ -\mathcal{G}_A^{(1\kappa)} \quad -\mathcal{G}_A^{(2\kappa)} \quad -\mathcal{G}_A^{(3\kappa)} \quad \dots \quad 0 \quad \dots \quad \mathcal{G}_A^{(\kappa N)} \right] \\
\dots &= \text{row} \left[ \dots \dots \dots \dots \dots \dots \dots \right] \\
\mathcal{L}_A^{(N)} &= \text{row} \left[ -\mathcal{G}_A^{(1N)} \quad -\mathcal{G}_A^{(2N)} \quad -\mathcal{G}_A^{(3N)} \quad \dots \quad -\mathcal{G}_A^{(\kappa N)} \quad \dots \quad 0 \right]
\end{aligned} \quad (\text{A.2})$$

With convention (3.15) the lower  $N$  rows of this table simplify to

$$\mathcal{L}_A^{(\mu)} = \text{row} [\dots, \hat{\mathcal{G}}_A^{(\mu\kappa)}, \dots], \quad 1 \leq \mu, \kappa \leq N. \quad (\text{A.3})$$

Table (A.2) enables to reduce the proof of Theorem 3.2 to an application of Theorem 2.4. Because of property (3.7) it is obvious that all 'components' of  $\mathcal{L}_A^{(\mu\star)}$ ,  $0 \leq \mu \leq N$ , are the *hermitean transposed* of the components of  $\mathcal{L}_A^{(\mu)}$ ,  $0 \leq \mu \leq N$ . Only for  $\mathcal{L}_A^{(\circ\star)}$  this is not immediately obvious. Let us check it in an *ad hoc* way by calculating the  $\kappa$ -th component of  $\mathcal{L}_A^{(\circ\star)}$ . In (A.1) replace  $\{\mathcal{A}_\mu^\dagger, \mathcal{A}_\nu^\dagger\}$  by the perturbation  $\{\mathcal{A}_\mu^\dagger + \varepsilon \delta_{\mu\kappa} H, \mathcal{A}_\nu^\dagger + \varepsilon \delta_{\nu\kappa} H\}$ . Now differentiate the result to  $\varepsilon$ . At  $\varepsilon = 0$  it becomes

$$\begin{aligned}
&\sum_{1 \leq \mu < \nu \leq N} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu\star)} : \{\delta_{\mu\kappa} H, \mathcal{A}_\nu^\dagger\} + \{\mathcal{A}_\mu^\dagger, \delta_{\nu\kappa} H\} \right] = \\
&= \sum_{\kappa < \nu \leq N} \text{Tr} \left[ \mathcal{G}_A^{(\kappa\nu\star)} : \{H, \mathcal{A}_\nu^\dagger\} \right] + \sum_{1 \leq \mu < \kappa} \text{Tr} \left[ \mathcal{G}_A^{(\mu\kappa\star)} : \{\mathcal{A}_\mu^\dagger, H\} \right] = \\
&= \sum_{\kappa < \nu \leq N} \text{Tr} \left[ \{\mathcal{A}_\nu^\dagger, \mathcal{G}_A^{(\kappa\nu\star)}\} : H \right] + \sum_{1 \leq \mu < \kappa} \text{Tr} \left[ \{\mathcal{G}_A^{(\mu\kappa\star)}, \mathcal{A}_\mu^\dagger\} : H \right] = \text{Tr} \left[ \sum_{\mu=1}^N \{\hat{\mathcal{G}}_A^{(\mu\kappa\star)}, \mathcal{A}_\mu^\dagger\} : H \right].
\end{aligned}$$

Finally one finds

$$\left[ \sum_{\mu=1}^N \{ \hat{\mathcal{G}}_A^{(\mu\kappa\star)}, \mathcal{A}_\mu^\dagger \} \right]^\dagger = - \sum_{\mu=1}^N \{ \hat{\mathcal{G}}_A^{(\mu\kappa)}, \mathcal{A}_\mu \}.$$

**Remark on Thm 4.9-b:** If it happens that

$$\begin{aligned} \mathcal{G}(\dots, e^{sS_\mu^\lambda} \partial_\lambda \mathcal{A}_\nu - e^{sS_\nu^\theta} \partial_\theta \mathcal{A}_\mu - \{ \mathcal{A}_\mu, \mathcal{A}_\nu \}, \dots; \dots, e^{sS_\mu^\lambda} \partial_\lambda \mathcal{A}_\nu^\dagger - e^{sS_\nu^\theta} \partial_\theta \mathcal{A}_\mu^\dagger + \{ \mathcal{A}_\mu^\dagger, \mathcal{A}_\nu^\dagger \}, \dots; \underline{x}) = \\ = \mathcal{G}(\dots, \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \{ \mathcal{A}_\mu, \mathcal{A}_\nu \}, \dots; \dots, \partial_\mu \mathcal{A}_\nu^\dagger - \partial_\nu \mathcal{A}_\mu^\dagger + \{ \mathcal{A}_\mu^\dagger, \mathcal{A}_\nu^\dagger \}, \dots; \underline{x}) + \mathcal{O}(s^2), \end{aligned}$$

it follows that

$$\text{Re} \sum_{\mu < \nu} \text{Tr} \left[ \mathcal{G}_A^{(\mu\nu)} : S_\mu^\lambda \partial_\lambda \mathcal{A}_\nu - S_\nu^\theta \partial_\theta \mathcal{A}_\mu \right] = 0.$$

## B Electromagnetism

Some more details on Example 3.4B:

$$\mathcal{G}_A = \sum_{0 \leq \mu < \nu \leq 3} (-1)^{\delta_{\mu 0} + \delta_{\nu 0}} \text{Tr} [\mathcal{F}_{\mu\nu}^\dagger \mathcal{F}_{\mu\nu}]$$

$$\mathcal{G}_A^{(01)} = -\mathcal{F}_{01}^\dagger \quad \mathcal{G}_A^{(02)} = -\mathcal{F}_{02}^\dagger \quad \mathcal{G}_A^{(03)} = -\mathcal{F}_{03}^\dagger \quad \mathcal{G}_A^{(12)} = \mathcal{F}_{12}^\dagger \quad \mathcal{G}_A^{(13)} = \mathcal{F}_{13}^\dagger \quad \mathcal{G}_A^{(23)} = \mathcal{F}_{23}^\dagger$$

Now (3.19) reads, for  $0 \leq \kappa \leq 3$ ,

$$\begin{aligned} \kappa = 0 : \quad & \partial_1 \mathcal{G}_A^{(01)} + \partial_2 \mathcal{G}_A^{(02)} + \partial_3 \mathcal{G}_A^{(03)} = \\ & = -\partial_1(\partial_0 \mathcal{A}_1^\dagger - \partial_1 \mathcal{A}_0^\dagger) - \partial_2(\partial_0 \mathcal{A}_2^\dagger - \partial_2 \mathcal{A}_0^\dagger) - \partial_3(\partial_0 \mathcal{A}_3^\dagger - \partial_3 \mathcal{A}_0^\dagger) \\ & = -\partial_0(\partial_1 \mathcal{A}_1^\dagger + \partial_2 \mathcal{A}_2^\dagger + \partial_3 \mathcal{A}_3^\dagger) + \partial_1 \partial_1 \mathcal{A}_0^\dagger + \partial_2 \partial_2 \mathcal{A}_0^\dagger + \partial_3 \partial_3 \mathcal{A}_0^\dagger \\ \kappa = 1 : \quad & -\partial_0 \mathcal{G}_A^{(01)} + \partial_2 \mathcal{G}_A^{(12)} + \partial_3 \mathcal{G}_A^{(13)} = \\ & = \partial_0(\partial_0 \mathcal{A}_1^\dagger - \partial_1 \mathcal{A}_0^\dagger) + \partial_2(\partial_1 \mathcal{A}_2^\dagger - \partial_2 \mathcal{A}_1^\dagger) + \partial_3(\partial_1 \mathcal{A}_3^\dagger - \partial_3 \mathcal{A}_1^\dagger) \\ & = \partial_0 \partial_0 \mathcal{A}_1^\dagger + \partial_1(-\partial_0 \mathcal{A}_0^\dagger + \partial_1 \mathcal{A}_1^\dagger + \partial_2 \mathcal{A}_2^\dagger + \partial_3 \mathcal{A}_3^\dagger) - (\partial_1 \partial_1 + \partial_2 \partial_2 + \partial_3 \partial_3) \mathcal{A}_1^\dagger \\ \kappa = 2 : \quad & -\partial_0 \mathcal{G}_A^{(02)} - \partial_1 \mathcal{G}_A^{(12)} + \partial_3 \mathcal{G}_A^{(23)} = \\ & = \partial_0(\partial_0 \mathcal{A}_2^\dagger - \partial_2 \mathcal{A}_0^\dagger) - \partial_1(\partial_1 \mathcal{A}_2^\dagger - \partial_2 \mathcal{A}_1^\dagger) + \partial_3(\partial_2 \mathcal{A}_3^\dagger - \partial_3 \mathcal{A}_2^\dagger) \\ & = \partial_0 \partial_0 \mathcal{A}_2^\dagger + \partial_2(-\partial_0 \mathcal{A}_0^\dagger + \partial_1 \mathcal{A}_1^\dagger + \partial_2 \mathcal{A}_2^\dagger + \partial_3 \mathcal{A}_3^\dagger) - (\partial_1 \partial_1 + \partial_2 \partial_2 + \partial_3 \partial_3) \mathcal{A}_2^\dagger \\ \kappa = 3 : \quad & -\partial_0 \mathcal{G}_A^{(03)} - \partial_1 \mathcal{G}_A^{(13)} - \partial_2 \mathcal{G}_A^{(23)} = \\ & = \partial_0(\partial_0 \mathcal{A}_3^\dagger - \partial_3 \mathcal{A}_0^\dagger) - \partial_1(\partial_1 \mathcal{A}_3^\dagger - \partial_3 \mathcal{A}_1^\dagger) - \partial_2(\partial_2 \mathcal{A}_3^\dagger - \partial_3 \mathcal{A}_2^\dagger) \\ & = \partial_0 \partial_0 \mathcal{A}_3^\dagger + \partial_3(-\partial_0 \mathcal{A}_0^\dagger + \partial_1 \mathcal{A}_1^\dagger + \partial_2 \mathcal{A}_2^\dagger + \partial_3 \mathcal{A}_3^\dagger) - (\partial_1 \partial_1 + \partial_2 \partial_2 + \partial_3 \partial_3) \mathcal{A}_3^\dagger \end{aligned}$$

If we put  $\mathcal{A}_0^\dagger = -\Phi$  and  $\text{col}[\mathcal{A}_1^\dagger, \mathcal{A}_2^\dagger, \mathcal{A}_3^\dagger] = \underline{A}$  we get Maxwell's equations 'in potential form'

$$\begin{cases} \frac{\partial}{\partial t} \text{div} \underline{A} + \Delta \Phi = 0 \\ \frac{\partial^2}{\partial t^2} \underline{A} - \Delta \underline{A} + \text{grad} \left( \frac{\partial}{\partial t} \Phi + \text{div} \underline{A} \right) = \underline{0} \end{cases} \quad (\text{B.1})$$

If the pair  $\underline{A}, \underline{B}$  satisfies this pair, then the pair  $\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \text{grad} \Phi$ ,  $\underline{B} = \text{rot} \underline{A}$ , satisfies the classical homogeneous Maxwell equations:

$$\partial_t \underline{B} = \text{rot} \partial_t \underline{A} = \text{rot}(-\underline{E} - \text{grad} \Phi) = -\text{rot} \underline{E}$$

$$\partial_t \underline{E} = \partial_t \partial_t \underline{A} - \text{grad} \partial_t \Phi = -\Delta \underline{A} + \text{grad} \text{div} \underline{A} = \text{rot} \text{rot} \underline{A} = \text{rot} \underline{B}$$

Finally, imposing the 'Lorenz-Gauge'  $\frac{\partial}{\partial t} \Phi + \text{div} \underline{A} = 0$ , we find the usual wave equations for  $\Phi$  and  $\underline{A}$ .

Any solution to the system (B.1) can be reduced to a solution which satisfies the Lorentz condition, by means of a 'gauge transform'  $\Phi \mapsto \Phi - \partial_t \Lambda$ ,  $\underline{A} \mapsto \underline{A} - \text{grad} \Lambda$ , leading to the same  $\underline{E}, \underline{B}$ -fields. cf. Jackson [J], p.241.

Similar results can be found for more general free fields governed by

$$\mathcal{G}_1 = \sum_{\mu\nu\theta^*\rho^*} g^{\mu\theta^*} g^{\nu\rho^*} \text{Tr} \left[ J \mathcal{F}_{\theta^*\rho^*}^\dagger J^{-1} \mathcal{F}_{\mu\nu} \right].$$

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